

Asymptotics of first-passage percolation on 1-dimensional graphs

Daniel Ahlberg*

July 13, 2011

Abstract

In this paper we consider standard first-passage percolation on certain 1-dimensional periodic graphs. One such graph of particular interest is the $\mathbb{Z} \times \{0, 1, \dots, K-1\}^{d-1}$ nearest neighbour graph for $d, K \geq 1$. Let $T(u, v)$ denote the time it takes for an infection started at u to reach v , and let $N(u, v)$ denote the length of the geodesic (path with minimal passage time) from u to v . We derive asymptotic results that show how the behaviour of first-passage percolation on 1-dimensional graphs differ from what is known or expected in higher dimensions. Let $\mathbf{n} = (n, 0, \dots, 0)$. By subadditivity $T(0, \mathbf{n})/n \rightarrow \mu$ for some $\mu > 0$ as $n \rightarrow \infty$, almost surely and in L^1 . We show that for some $\sigma > 0$, as $n \rightarrow \infty$, $(T(0, \mathbf{n}) - \mu n)/\sigma\sqrt{n}$ converges in distribution to a standard normal, and moreover, that $\limsup_{n \rightarrow \infty} (T(0, \mathbf{n}) - \mu n)/\sigma\sqrt{2n \log \log n} = 1$, almost surely. We further prove that $\mathbb{E}[T(0, \mathbf{n})]$ and $\text{Var}(T(0, \mathbf{n}))$ are monotonic in n , for large enough n . Results for $N(0, \mathbf{n})$ corresponding to the results mentioned for $T(0, \mathbf{n})$ are also derived.

We also allow different sets of initially infected vertices, and construct an exact coupling of two infections with different starting configurations. Using this coupling we prove a 0–1 law.

1 Introduction

First-passage percolation was first considered by Hammersley and Welsh (1965). It can be thought of as a model for the spread of an infection on a connected graph with set of vertices \mathbb{V} and set of edges \mathbb{E} . Associate to the edges of the graph non-negative i.i.d. random variables $\{\tau_e\}_{e \in \mathbb{E}}$, referred to as *passage times*. We will denote the passage time distribution by $P_\tau(\cdot) := P(\tau_e \in \cdot)$. To avoid trivialities, we assume throughout this paper that P_τ does not concentrate all mass at a single point. With the present interpretation of the model, the passage time of an edge should be thought of as the random time it takes for an infection to spread along the edge. Consider the process where we start with a finite set $I \subset \mathbb{V}$ of infected vertices. As time starts, the infection spreads to adjacent vertices with delays indicated by the passage times.

Let us by a *path* refer to an alternating sequence of vertices and edges; $v_0, e_1, v_1, \dots, e_m, v_m$, beginning and ending with a vertex, such that v_k is the endpoint of the edges e_k and e_{k+1} that precedes and follows v_k . The vertices v_0 and v_m are referred to as endpoints of the path. A path with one endpoint in U and the other in V , where $U, V \subset \mathbb{V}$, will be referred to as a

*Department of Mathematical Sciences, University of Gothenburg, and Department of Mathematical Sciences, Chalmers University of Technology. E-mail: md1ahlbda@chalmers.se

path from U to V . We will repeatedly abuse notation and identify a path with its set of edges, and occasionally with its set of vertices. For a path Γ , we define the passage time of Γ as $T(\Gamma) := \sum_{e \in \Gamma} \tau_e$, and define the *passage time*, or *first-passage time*, between two sets of vertices $U, V \subset \mathbb{V}$ as

$$T(U, V) := \inf\{T(\Gamma) : \Gamma \text{ is a path from } U \text{ to } V\}.$$

We are often interested in the case when $U = \{u\}$ or $V = \{v\}$. We will in such case simply write $T(u, v)$ for $T(\{u\}, \{v\})$. The main features of first-passage percolation are retained in

$$T(v) := T(I, v)$$

interpreted as the time it takes for the infection started in I to reach the vertex v , and

$$B_t := \{v \in \mathbb{V} : T(v) \leq t\},$$

the set of infected vertices at time t .

A typical choice for the underlying graph is the usual \mathbb{Z}^d *lattice*, whose vertices are the elements of \mathbb{Z}^d , and where two vertices are connected with an edge if their Euclidean distance is one. In this paper, though, we will consider first-passage percolation on 1-dimensional graphs. However, we begin with a presentation of some of the results for first-passage percolation on the \mathbb{Z}^d lattice. Thereafter, the motivation for considering 1-dimensional graphs, as well as our results themselves, will be better understood. A more detailed survey of first-passage percolation can be found in Howard (2004).

It is customary to consider first-passage percolation with a single initially infected vertex at the origin. However, we have reasons to be interested in different initial configurations of the infection. The results we are about to review regarding the \mathbb{Z}^d lattice hold for any finite initially infected set.

A challenging task, already considered by Hammersley and Welsh (1965), is to describe the behaviour of $T(v)$ when $|v|$ is large. It follows from its definition that $T(u, v)$ is subadditive, i.e.,

$$T(u, v) \leq T(u, w) + T(w, v)$$

for any vertices u, v and w in \mathbb{Z}^d . Let

$$Y = \min(\tau_1, \dots, \tau_{2d}), \tag{1.1}$$

where τ_1, \dots, τ_{2d} are independent and distributed according to P_τ . Thus, if $E[Y] < \infty$, Kingman's Subadditive ergodic theorem says that there is a constant $\mu(\mathbf{e}_1)$, referred to as the *time constant*, such that

$$\lim_{n \rightarrow \infty} \frac{T(\mathbf{n})}{n} = \mu(\mathbf{e}_1), \quad \text{almost surely and in } L^1, \tag{1.2}$$

where $\mathbf{e}_1 = (1, 0, \dots, 0)$, and $\mathbf{n} = n\mathbf{e}_1$. The same holds in every direction. Let $\bar{x} \in \mathbb{R}^d$ be such that $|\bar{x}| = 1$. If $\lfloor n\bar{x} \rfloor$ denotes the coordinate-wise integer part of $n\bar{x}$, then there is a $\mu(\bar{x})$ such that

$$\lim_{n \rightarrow \infty} \frac{T(\lfloor n\bar{x} \rfloor)}{n} = \mu(\bar{x}), \quad \text{almost surely and in } L^1.$$

In fact, one can say more about this asymptotic growth. If we consider B_t , we can state results about the growth in all directions simultaneously. A first such result was due to Richardson (1973). For convenience, we replace B_t by the subset of \mathbb{R}^d defined as

$$\tilde{B}_t := \{x \in \mathbb{R}^d : x \in v + [0, 1]^d \text{ for some } v \in B_t\}, \quad (1.3)$$

The following version of Richardson's result is due to Cox and Durrett (1981), and states that the set of infected vertices grows linearly with t and has a nonrandom asymptotic shape.

Theorem 1.1 (Shape theorem). *Consider first-passage percolation on \mathbb{Z}^d with i.i.d. passage times such that*

$$\mathbb{E} [Y^d] < \infty, \quad (1.4)$$

for Y defined as in (1.1). If $\mu(\mathbf{e}_1) > 0$, then there exists a nonrandom, compact, convex subset B^ in \mathbb{R}^d with nonempty interior such that for all $\epsilon > 0$, almost surely,*

$$(1 - \epsilon)B^* \subset \frac{1}{t}\tilde{B}_t \subset (1 + \epsilon)B^*, \quad \text{for } t \text{ large enough.}$$

If $\mu(\mathbf{e}_1) = 0$, then for every compact set K in \mathbb{R}^d , almost surely,

$$K \subset \frac{1}{t}\tilde{B}_t, \quad \text{for } t \text{ large enough.}$$

In addition, it was shown by Kesten (1986) that

$$\mu(\mathbf{e}_1) = 0 \text{ if and only if } P_\tau(\{0\}) \geq p_c(d),$$

where $p_c(d)$ is the critical value for independent bond percolation on the \mathbb{Z}^d lattice. An elementary argument shows that $\mathbb{E}[\tau_e^2] < \infty$ is sufficient for (1.4) to hold.

As the Shape theorem establishes a law of large numbers for the sequence $T(\lfloor n\bar{x} \rfloor)$, it is natural to ask about the fluctuations of the same sequence. They have turned out to be harder to understand, and depend on the dimension d . For $d = 1$, $T(n)$ reduces to a sum of i.i.d. random variables, from which it is immediate that

$$\text{Var}(T(n)) = n \text{Var}(\tau_e).$$

Kesten (1993) showed that for any $d \geq 1$, if $P_\tau(\{0\}) < p_c$ and $\mathbb{E}[\tau_e^2] < \infty$, then there are constants $C_1 > 0$ and $C_2 < \infty$ such that

$$C_1 \leq \text{Var}(T(\mathbf{n})) \leq C_2 n, \quad \text{for all } n \geq 1.$$

More precise results have been few. Benjamini, Kalai and Schramm (2003) gave an example which showed that for first-passage percolation on \mathbb{Z}^d for $d \geq 2$, with $\{a, b\}$ -valued passage times, where $0 < a < b < \infty$, there is a constant C such that

$$\text{Var}(T(\mathbf{n})) \leq C \frac{n}{\log n}, \quad \text{for all } n \geq 2. \quad (1.5)$$

This result was later extended by Benaïm and Rossignol (2006, 2008) to include a wider class of passage time distributions. This is still far from what is believed to be the precise growth rate of $\text{Var}(T(\mathbf{n}))$. For $d = 2$ it is believed that $\text{Var}(T(\mathbf{n}))$ is of the order $n^{2/3}$, and it is not clear which behaviour to expect in higher dimensions (see Newman and Piza (1995); Benjamini et al. (2003) for short resumés). For $d = 2$ Newman and Piza (1995) have shown, given that the passage-time distribution does not have a too big point mass at $\inf\{x \geq 0 : P_\tau([0, x]) > 0\}$, that there is a constant $C > 0$ such that

$$\text{Var}(T(\mathbf{n})) \geq C \log n,$$

for all $n \geq 1$. The same lower bound was found independently by Pemantle and Peres (1994), in the case of exponential passage times.

Classical limit theorems on 1-dimensional graphs. In this paper we consider first-passage percolation on essentially 1-dimensional periodic graphs defined as follows.

Definition 1.2. *The class of essentially 1-dimensional periodic graphs consists of all connected graphs \mathcal{G} that can be constructed in the following manner. Let $\{\mathcal{G}_n\}_{n \in \mathbb{Z}}$ be a sequence of identical copies of some finite connected deterministic graph, each with set of vertices $\mathbb{V}_{\mathcal{G}_n} = \{v_{n,1}, \dots, v_{n,K}\}$ and set of edges $\mathbb{E}_{\mathcal{G}_n} = \{e_{n,1}, \dots, e_{n,l}\}$. Fix a nonempty set $J \subseteq \{(i, j) : 1 \leq i, j \leq K\}$, and connect \mathcal{G}_n to \mathcal{G}_{n+1} for each n by adding an edge $e(v_{n,i}, v_{n+1,j})$ between $v_{n,i}$ and $v_{n+1,j}$, for each $(i, j) \in J$. Let $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ denote the resulting graph, where*

$$\mathbb{V} = \bigcup_{n \in \mathbb{Z}} \mathbb{V}_{\mathcal{G}_n} \quad \text{and} \quad \mathbb{E} = \bigcup_{n \in \mathbb{Z}} (\mathbb{E}_{\mathcal{G}_n} \cup \{e(v_{n,i}, v_{n+1,j}) : (i, j) \in J\}).$$

We will write $\mathbb{E}_{\mathcal{G}_n}^*$ for $\mathbb{E}_{\mathcal{G}_n} \cup \{e(v_{n,i}, v_{n+1,j}) : (i, j) \in J\}$, and say that a vertex v of \mathcal{G} is at level n if $v \in \mathbb{V}_{\mathcal{G}_n}$.

An essentially 1-dimensional graph of particular interest is the $\mathbb{Z} \times \{0, 1, \dots, K-1\}^{d-1}$ nearest neighbour graph, i.e., the sub-graph of the \mathbb{Z}^d lattice which has set of vertices $\mathbb{Z} \times \{0, 1, \dots, K-1\}^{d-1}$ for some $d, K \geq 1$, and where any two vertices are connected by an edge if their Euclidean distance is 1. We will refer to this graph as the (K, d) -tube (cf. Figure 1). We can think of

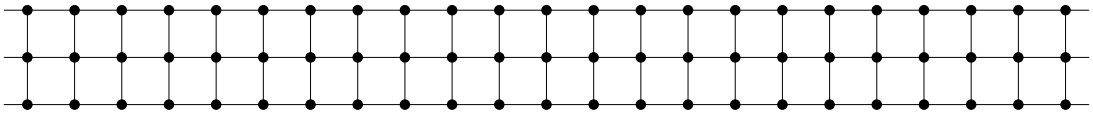


Figure 1: A part of the $(3, 2)$ -tube.

the (K, d) -tube as the essentially 1-dimensional periodic graph constructed from a sequence of graphs with vertex set $\mathbb{V}_{\mathcal{G}_n} = \{n\} \times \{0, 1, \dots, K-1\}^{d-1}$ and where any two vertices at Euclidean distance one are joined by an edge. With this construction, the vertices at level n are exactly the ones with first coordinate n .

Because of the unspecified structure of the underlying graph, it is convenient to consider

$$T_n := T(I, \mathbb{V}_{\mathcal{G}_n}), \tag{1.6}$$

interpreted as the time until a vertex at level n is infected. To consider T_n is natural, but is in no way necessary for the results we obtain. In fact, we shall see that the asymptotic behaviour of the sequence $\{T_n\}_{n \geq 1}$ is the same as that for the sequence $\{\max_{v \in \mathbb{V}_{\mathcal{G}_n}} T(v)\}_{n \geq 1}$, and the sequence $\{T(v_n)\}_{n \geq 1}$, where $\{v_n\}_{n \geq 1}$ is any sequence of vertices such that v_n is at level n . We will for that reason let $\{\hat{T}_n\}_{n \geq 1}$ denote any of the three sequences above, and state several of our results for \hat{T}_n . It will then be understood that the result holds for any of the three sequences.

Our main results concerns first-passage percolation on any essentially 1-dimensional periodic graph \mathcal{G} , with passage-time distribution that does not concentrate all mass at a single point. We will prove that there are non-negative, finite constants $\mu = \mu(\mathcal{G})$ and $\sigma = \sigma(\mathcal{G})$, such that the following holds.

Theorem 1.3 (Law of large numbers). *If $E[\tau_e] < \infty$, then*

$$\lim_{n \rightarrow \infty} \frac{\hat{T}_n}{n} = \mu, \quad \text{almost surely.} \quad (1.7)$$

If $E[\tau_e^r] < \infty$ for some $r \geq 1$, then

$$\left\{ \left(\hat{T}_n/n \right)^r \right\}_{n \geq 1} \quad \text{is uniformly integrable,}$$

and the convergence of (1.7) holds also in L^r .

Theorem 1.4 (Central limit theorem). *If $E[\tau_e^2] < \infty$, then*

$$\frac{\hat{T}_n - \mu n}{\sigma \sqrt{n}} \xrightarrow{d} \chi, \quad \text{in distribution,}$$

as $n \rightarrow \infty$, where χ has a standard normal distribution.

Let $\mathcal{L}(\{x_n\}_{n \geq 1})$ denote the set of limit points of a real-valued sequence $\{x_n\}_{n \geq 1}$.

Theorem 1.5 (Law of the iterated logarithm). *If $E[\tau_e^2] < \infty$, then*

$$\mathcal{L} \left(\left\{ \frac{\hat{T}_n - \mu n}{\sigma \sqrt{2n \log \log n}} \right\}_{n \geq 3} \right) = [-1, 1], \quad \text{almost surely.}$$

In particular, almost surely,

$$\limsup_{n \rightarrow \infty} \frac{\hat{T}_n - \mu n}{\sigma \sqrt{2n \log \log n}} = 1, \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{\hat{T}_n - \mu n}{\sigma \sqrt{2n \log \log n}} = -1.$$

Note that the almost sure and L^1 -convergence in Theorem 1.3 actually follows from Kingman's Subadditive ergodic theorem. However, it is for the understanding of our approach instructive to state and reprove it, as we do. As a consequence of the regenerative structure explored in Section 2, μ and σ will be given by explicit formulas. For this reason, it will become clear that μ and σ are non-negative, finite, and depend on the underlying graph \mathcal{G} and the passage time distribution, but do not depend on the set of initially infected vertices I , nor on which

of the three sequences considered that $\{\hat{T}_n\}_{n \geq 1}$ may represent. We will see (in Proposition 5.7) that μ and σ varies continuously with respect to P_τ . We preferred at this stage to give simple moment conditions in Theorems 1.3, 1.4 and 1.5. But we will later point out that they may in fact be relaxed somewhat (cf. Remark 3.4).

At a comparison with the asymptotic results in higher dimensions, Theorem 1.3 is the 1-dimensional analogue to the Shape theorem. Theorems 1.4 and 1.5 on the other hand, point out a 1-dimensional behaviour that is not expected in higher dimensions. In particular, $\text{Var}(T_n)$ grows linearly in n , in contrast to the higher dimensional sub-diffusive behaviour in (1.5), pointed out by Benjamini et al. (2003). However, we should also mention a result by Kesten and Zhang (1997) when $d = 2$ and $P_\tau(\{0\}) = p_c(2) = 1/2$. They have showed that $T(\mathbf{n}) - \mathbb{E}[T(\mathbf{n})]$ converges to a standard normal distribution, when scaled appropriately. This case is considered critical, and the scaling factor is known to grow of order $\log n$.

The classical Central limit theorem for i.i.d. sequences extends to a functional central limit theorem, known as *Donsker's theorem*. In contrast to the classical Central limit theorem that treats weak convergence of real-valued random variables, Donsker's theorem treats weak convergence of real-valued random functions. Theorem 1.4 also extends to a functional version, with the same limiting distribution as the regular Donsker theorem, i.e., Wiener measure. For the precise statement and a proof, see Theorem 3.6.

We should at this point mention a related, but independent, work by Chatterjee and Dey (2009). They consider first-passage percolation on nearest neighbour graphs of the form $\mathbb{Z} \times \{-K, \dots, K\}^{d-1}$. In our terminology, this is precisely the $(2K + 1, d)$ -tube. Introduce the notation $a_n(K)$ for the passage time $T(\mathbf{0}, \mathbf{n})$ between the origin and \mathbf{n} on that graph. Their main result essentially says that if $\mathbb{E}[\tau_e^r] < \infty$ for some $r > 2$, then there exists $\alpha = \alpha(d, r)$ such that if $K_n = o(n^\alpha)$, then

$$\frac{a_n(K_n) - \mathbb{E}[a_n(K_n)]}{\sqrt{\text{Var}(a_n(K_n))}} \xrightarrow{d} \chi, \quad \text{in distribution,} \quad (1.8)$$

as $n \rightarrow \infty$, where χ has a standard normal distribution. When $\mathbb{E}[\tau_e^r] < \infty$ for all $r \geq 1$, then $\alpha < 1/(d+1)$ is sufficient for (1.8) to hold. This result is similar to our Theorem 1.4, and applies to cases that Theorem 1.4 does not. The method of proof used in Chatterjee and Dey (2009) is different from ours, and we note that they require a slightly stronger moment condition than we do with our techniques (see also Remark 3.4). In Chatterjee and Dey (2009), (1.8) is also extended to hold for graphs of the form $\mathbb{Z} \times \mathcal{G}$, which is a subclass to the class of essentially 1-dimensional periodic graphs defined in Definition 1.2. Moreover, (1.8) extends to a functional central limit theorem similar to our Theorem 3.6. Again here, Chatterjee and Dey require that $\mathbb{E}[\tau_e^r] < \infty$ for some $r > 2$ in order for the functional limit theorem to hold. We emphasise that the present work was prepared simultaneously and independently of the work by Chatterjee and Dey (2009) by methods distinct from those in their paper. There seem to be advantages with the techniques used in this paper, as well as with the techniques used by Chatterjee and Dey. To further exclude questions of originality, we also mention that this paper is an extended version of the earlier manuscript Ahlberg (2008), in which several of the results presented here were included, among them Theorem 1.4.

Monotonicity of mean and variance. It seems natural to believe that the mean and variance of $T(u, v)$ increase with the distance between u and v . We will prove two theorems concerning this.

Theorem 1.6. *Let $v_{n,i}$ denote a specific vertex at level n . For all $i = 1, \dots, K$, if $E[\tau_e] < \infty$, then for some $C_i \in \mathbb{R}$, as $n \rightarrow \infty$,*

$$E[T(v_{n,i})] = \mu n + C_i + o(1).$$

A direct consequence of this result is that

$$E[T(v_{n+1,i}) - T(v_{n,i})] \rightarrow \mu, \quad \text{as } n \rightarrow \infty.$$

Since $\mu > 0$, this proves monotonicity of $E[T(v_{n,i})]$, for large n . This question dates back to Hammersley and Welsh (1965). That we have such monotonicity on \mathbb{Z} is completely trivial, but on \mathbb{Z}^d for $d \geq 2$ it is still an open problem to solve. A counterexample given by van den Berg (1983) shows that such monotonicity result for the expected travel time from $(0, 0)$ to $(n, 0)$ on the $\{0, 1, \dots, n\} \times \mathbb{Z}$ nearest neighbour graph does not hold for every n . This indicates that it might not be possible to extend Theorem 1.6 to say that the mean travel time is monotonous for *all* n . The same remark should concern also the following result which proves monotonicity of the variance of the travel time for large n .

Theorem 1.7. *Let $v_{n,i}$ denote a specific vertex at level n . For all $i = 1, \dots, K$, if $E[\tau_e^2] < \infty$, then for some $C_i \in \mathbb{R}$, as $n \rightarrow \infty$,*

$$\text{Var}(T(v_{n,i})) = \sigma^2 n + C_i + o(1).$$

Asymptotic behaviour of geodesics. First-passage percolation offers more than describing the behaviour of the passage time between vertices. One matter which has received a lot of attention is along which edges (path) an infection travels from one vertex to another. Do such paths exist, and if they do, how do they behave? On the \mathbb{Z}^d lattice such minimising paths are known to exist for passage-time distributions with not too big point mass at zero (see Howard (2004) for more precision of the statement). A simple argument that shows that such paths exist on essentially 1-dimensional periodic graphs (for any passage-time distribution) is given in Proposition 5.1. As customary, we will use the term *geodesic* to refer to a path $\gamma(u, v)$ attaining the minimal passage time, i.e., such that $T(\gamma(u, v)) = T(u, v)$. Geodesics are not necessarily unique when the passage-time distribution has atoms (for continuous distributions they are; cf. Proposition 5.1). For this reason, fix a deterministic rule to choose one when several are possible (e.g. the shortest, with some additional rule for breaking ties).

Let N_n and $N(v)$ denote the length of the geodesic realising T_n and $T(v)$, respectively. Let $\{\hat{N}_n\}_{n \geq 1}$ denote either of the sequences $\{N_n\}_{n \geq 1}$, $\{\max_{v \in \mathbb{V}_{G_n}} N(v)\}_{n \geq 1}$ and $\{N(v_n)\}_{n \geq 1}$, where $\{v_n\}_{n \geq 1}$ is any sequence of vertices such that v_n is at level n . We state here the following result, and refer the reader to Section 5 and Theorem 5.2 and 5.3, for additional result concerning asymptotics of length of geodesics.

Theorem 1.8. *There is a finite constant α such that, for any $r \geq 1$,*

$$\lim_{n \rightarrow \infty} \frac{\hat{N}_n}{n} = \alpha, \quad \text{almost surely and in } L^r.$$

On the \mathbb{Z}^2 lattice, Zhang and Zhang (1984) showed that a similar strong law, as the one exhibited in the above theorem, holds for "supercritical" passage-time distributions, i.e., passage-time distributions such that $P_\tau(\{0\}) > 1/2$. Moreover, Garet and Marchand (2004, 2007) have considered the related case of first-passage percolation on the \mathbb{Z}^d lattice with passage times distributed as $P_\tau = p\delta_1 + (1-p)\delta_\infty$, for some $p > p_c(d)$. In this situation the length of the geodesic between two vertices equals the passage time between them (given that it is finite). Assume that the origin lies in the unique infinite cluster, and for $z \in \mathbb{Z}^d$ let $\{u_{n,z}\}_{n \geq 1}$ denotes the subsequence of $\{n\}_{n \geq 1}$ such that $u_{n,z}z$ lies in the infinite cluster. They showed that, almost surely,

$$\exists \lim_{n \rightarrow \infty} \frac{T(0, u_{n,z}z)}{u_{n,z}}, \quad \text{uniformly in } z \in \mathbb{Z}^d.$$

They further prove exponential decay of deviations away from this limit. Whether the same limiting behaviour, as in the above theorem, holds for general passage-time distributions on the \mathbb{Z}^d lattice is not known (see Howard (2004) for further reference).

The (K, d) -tube case. First-passage percolation on (K, d) -tubes is of particular interest, since it can be compared in a natural way to first-passage percolation on the \mathbb{Z}^d lattice. As an example of such a comparison, we can see how Theorem 1.3 is a 1-dimensional analogue to the Shape theorem. Replace B_t with the set \tilde{B}_t as in (1.3). Let μ_K denote the time constant of Theorem 1.3 associated with the (K, d) -tube, and set

$$B^* = B^*(t) = [-\mu_K^{-1}, \mu_K^{-1}] \times [0, K/t]^{d-1}.$$

The almost sure convergence in Theorem 1.3 is then equivalent to that for all $\epsilon > 0$, almost surely,

$$(1 - \epsilon)B^* \subset \frac{1}{t}\tilde{B}_t \subset (1 + \epsilon)B^*, \quad \text{for large } t. \quad (1.9)$$

We can in fact allow ϵ to tend to zero with t . The precise size of the fluctuations in (1.9) follows from Theorem 1.5. We refer the reader to Corollary 3.5 for the precise statement.

Let μ_K denote the time constant associated with the (K, d) -tube (for fixed d). A simple coupling argument shows that $\mu_{K+1} \leq \mu_K$. In fact strict inequality holds for all $K \geq 1$ (cf. Proposition 5.10). Apart from being decreasing, the sequence $\{\mu_K\}_{K \geq 1}$ is bounded below by $\mu(\mathbf{e}_1)$. Thus, the sequence is convergent. In Proposition 5.11 we prove that

$$\lim_{K \rightarrow \infty} \mu_K = \mu(\mathbf{e}_1).$$

This shows that the rate of growth of an infection on the (K, d) -tube approaches the rate of growth of an infection on the \mathbb{Z}^d lattice, as K increases. Does the same monotonic behaviour hold for the constants σ_K^2 and α_K , that appear in Theorem 1.4 and 1.8, associated with the (K, d) -tube? There is no argument known to us that implies monotonicity. In view of the belief of the fluctuations in higher dimensions, and the sub-diffusive behaviour shown in (1.5), it seems reasonable to believe that σ_K^2 tends to zero as $K \rightarrow \infty$.

Coupling and a 0–1 law. Another main part of this paper consists of the construction of a coupling of two first-passage percolation infections. As an application of the coupling we prove a 0–1 law. Define the σ -algebra $\mathcal{T}_t := \sigma(\{B_s\}_{s \geq t})$ and the tail σ -algebra $\mathcal{T} := \cap_{t \geq 0} \mathcal{T}_t$. We may think of \mathcal{T}_t as the σ -algebra of events that do not depend on the times at which vertices are infected before time t .

Theorem 1.9 (0–1 law). *Consider first-passage percolation on an essentially 1-dimensional periodic graph \mathcal{G} , with a finite set of initially infected vertices. Assume that the passage time distribution has an absolutely continuous component (with respect to Lebesgue measure). Then $P(A) \in \{0, 1\}$, for any event $A \in \mathcal{T}$.*

The 0–1 law follows from an application of the following coupling.

Proposition 1.10 (Coupling). *Let I and I' be finite subsets of the set of vertices of an essentially 1-dimensional periodic graph \mathcal{G} . Assume that the passage time distribution P_τ has an absolutely continuous component (with respect to Lebesgue measure). There exists a coupling of $\{\tau_e\}_{e \in \mathbb{E}}$ and $\{\tau'_e\}_{e \in \mathbb{E}}$ such that $\{\tau_e\}_{e \in \mathbb{E}}$ and $\{\tau'_e\}_{e \in \mathbb{E}}$ form sequences of i.i.d. random variables with distribution P_τ , and if first-passage percolation is performed with $(I, \{\tau_e\}_{e \in \mathbb{E}})$ and $(I', \{\tau'_e\}_{e \in \mathbb{E}})$, respectively, then with probability one there exists $T_c < \infty$, such that*

$$B_t = B'_t, \quad \text{for all } t \geq T_c.$$

A similar coupling is presented also for discrete passage time distributions, but then on the more restrictive class of (K, d) -tubes (cf. Proposition 6.2). Theorem 1.9 is extended to include this case as well. Motivating examples are given to show why it is not possible to make the coupling as general as Proposition 1.10 also in the discrete case (cf. Remark 6.6 and 6.7).

The mild condition of an absolutely continuous component to be sufficient for the 0–1 law on essentially 1-dimensional periodic graphs, opens up for a discussion. We do not know on which other graphs this condition is sufficient. But, we give an example showing that a 0–1 law analogous to Theorem 1.9 cannot hold on the binary tree \mathbb{T}^2 . An interesting and challenging case to settle would be on the \mathbb{Z}^d lattice.

The main results of this paper will be based on a “regenerative” nature that arises for first-passage percolation on essentially 1-dimensional periodic graphs. What we mean by a regenerative behaviour will be clarified in the next section, where we also derive the properties of the regenerative structure that will recur throughout this paper. As will become apparent, the idea is to identify a suitable regenerative sequence (cf. Definition 2.1). How the regenerative behaviour arises naturally for exponentially distributed passage times is illustrated in Section 2.1. The general case is thereafter treated in detail in Section 2.2.

Once the regenerative behaviour is understood, some of the results we provide will follow, either from simple arguments or from already known results. Other results that we provide do not follow as easily, and will require an essential amount of additional work. This will be emphasised in connection with their proofs. In Section 3, the regenerative behaviour is used to prove Theorems 1.3, 1.4 and 1.5, among others. Monotonicity of mean and variance of the travel time, i.e., Theorem 1.6 and 1.7, is proved in Section 4. Section 5 is dedicated to study geodesics and properties of μ , σ and α . In the final Section 6 the coupling of Proposition 1.10 is

constructed, in its continuous and its discrete version. The 0–1 law Theorem 1.9 is also derived and the counterexample to the 0–1 law on trees is presented at the very end.

2 Regenerative behaviour

Definition 2.1. *We say that a sequence $\{X_k\}_{k \geq 1}$ of random variables is a regenerative sequence if there exists an increasing sequence of random variables $\{\lambda_k\}_{k \geq 0}$ such that*

- a) $\{\lambda_k - \lambda_{k-1}\}_{k \geq 1}$ forms an i.i.d. sequence, and
- b) $\{X_{\lambda_k} - X_{\lambda_{k-1}}\}_{k \geq 1}$ forms a sequence of i.i.d. non-negative random variables.

We will refer to $\{\lambda_k\}_{k \geq 0}$ as the sequence of regenerative levels.

Some readers may recognise the sub-sequence $\{X_{\lambda_k}\}_{k \geq 0}$ as a renewal sequence, and the sequence $\{(X_{\lambda_k}, \lambda_k)\}_{k \geq 0}$ as a 2-dimensional renewal sequence.

The idea of how to identify a suitable regenerative sequence arises naturally for first-passage percolation with exponentially distributed passage times. We begin with an illustration of this on the $(2, 2)$ -tube. In Section 2.2 we will generalise this idea to concern general passage time distributions, and any essentially 1-dimensional periodic graph.

2.1 Exponential passage times

Let the edges of the $(2, 2)$ -tube be equipped with i.i.d. exponential passage times $\{\tau_e\}_{e \in \mathbb{E}}$, and let both vertices at level zero be initially infected. At any fixed time t , given the infected component B_t , each edge with exactly one endpoint in the infected component is equally likely to be passed by the infection next. Thus, at each level, with probability at least $1/2$, both vertices will become infected before any vertex at the following level. It follows that with probability one, at some level r , both vertices will become infected before any vertex at level $r + 1$. Denote by ρ the first level for which this happens, and let τ_ρ denote the time at which this happens. By the lack-of-memory property, the time it takes for the infection from this moment to reach m levels further has the same distribution as the time it would take to reach level m , i.e.,

$$T_{\rho+m} - \tau_\rho \stackrel{d}{=} T_m. \quad (2.1)$$

In fact, at infinitely many levels, both vertices at that level will be infected before any vertex at higher levels. If we repeat the argument, we generate a sequence of (regenerative) levels $\{\rho_k\}_{k \geq 1}$ (see Figure 2), with corresponding sequence of instants $\{\tau_{\rho_k}\}_{k \geq 1}$, such that (2.1) holds. Since the passage times are i.i.d., the consecutive differences $\rho_{k+1} - \rho_k$ will be i.i.d., as well as the differences $\tau_{\rho_{k+1}} - \tau_{\rho_k}$. It follows that $\{\max_{v \in \mathbb{V}_{\mathcal{G}_n}} T(v)\}_{n \geq 1}$ is a regenerative sequence.

The point of the regenerative sequence is the following. Note that the n th (regenerative) level and the time at which it occurs may be written as sums of i.i.d. random variables, i.e.,

$$\rho_n = \sum_{k=0}^{n-1} \rho_{k+1} - \rho_k \quad \text{and} \quad \tau_{\rho_n} = \sum_{k=0}^{n-1} \tau_{\rho_{k+1}} - \tau_{\rho_k},$$

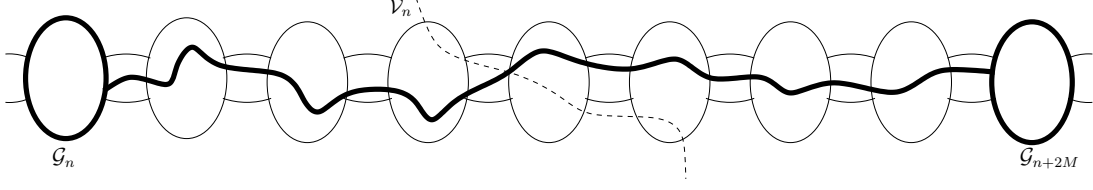


Figure 3: The graph \mathcal{G} between level n and $n + 2M$. If A_n occurs, the thick edges at level n , level $n + 2M$ and of the path γ_n are “quick”.

Lemma 2.3. *Let t' and t'' be constants such that $m_\tau < t' < t'' < M_\tau$. Then there exists $M \in \mathbb{N}$, such that:*

- a) *If A_n occurs, then for all $u \in \bigcup_{k \leq n} \mathbb{V}_{\mathcal{G}_k}$ and $v \in \bigcup_{k \geq n+2M} \mathbb{V}_{\mathcal{G}_k}$*

$$T(u, v) = T(u, \hat{v}_{n+M}) + T(\hat{v}_{n+M}, v), \quad (2.5)$$

and $T(\Gamma) > T(u, v)$ for any path Γ between u and v that does not visit \hat{v}_{n+M} .

- b) *$T(u, \hat{v}_{n+M})$ and $T(\hat{v}_{n+M}, v)$ are conditionally independent given A_n . In addition, given A_n , $T(u, \hat{v}_{n+M})$ is conditionally independent of the passage time of any edge beyond level $n + 2M$, and $T(\hat{v}_{n+M}, v)$ is conditionally independent of the passage time of any edge before level n .*

Proof. It suffice to prove the lemma for $u \in \mathbb{V}_{\mathcal{G}_n}$ and $v \in \mathbb{V}_{\mathcal{G}_{n+2M}}$. For given t' and t'' , choose

$$M > \frac{|\mathbb{E}_{\mathcal{G}_n}|t'}{t'' - t'},$$

where $|\cdot|$ denotes the cardinality of the set. Set $\beta := \text{dist}(\hat{v}_{n+M}, \mathbb{V}_{\mathcal{G}_{n+2M}})$, where $\text{dist}(v, V)$ denotes the smallest number of edges one has to pass in order to reach a vertex of V from v , and define (see Figure 3)

$$\mathcal{V}_n := \left\{ v \in \bigcup_{j=n}^{n+2M} \mathbb{V}_{\mathcal{G}_j} : \text{dist}(v, \mathbb{V}_{\mathcal{G}_{n+2M}}) = \beta \right\}.$$

We will prove that, given A_n ,

$$T(u, \hat{v}_{n+M}) < T(u, w) \quad \text{and} \quad T(\hat{v}_{n+M}, v) < T(w, v) \quad (2.6)$$

for all $w \in \mathcal{V}_n \setminus \{\hat{v}_{n+M}\}$. This proves that $T(\Gamma) > T(u, v)$ for all paths Γ between u and v that does not visit \hat{v}_{n+M} , since each path from u to v has to pass some vertex in \mathcal{V}_n . Thus, also (2.5) holds. That $T(u, \hat{v}_{n+M})$ and $T(\hat{v}_{n+M}, v)$ are conditionally independent given A_n is easily seen from the following observation. When A_n occurs, it follows from (2.6) that $T(u, \hat{v}_{n+M})$ is the infimum of $T(\Gamma)$ over all paths Γ from u to \hat{v}_{n+M} that intersects \mathcal{V}_n only in \hat{v}_{n+M} , whereas $T(\hat{v}_{n+M}, v)$ is the infimum of $T(\Gamma)$ over all paths Γ from \hat{v}_{n+M} to v that intersects \mathcal{V}_n only in

\hat{v}_{n+M} . Hence, the infima of passage times are taken over paths in disjoint parts of the graph. The remaining statement in *b*) follows similarly.

To deduce (2.6), condition on A_n . By definition of γ_n and \mathcal{V}_n ,

$$T(w', \hat{v}_{n+M}) < T(w', w)$$

for any vertex w' visited by γ_n , and $w \in \mathcal{V}_n \setminus \{\hat{v}_{n+M}\}$. Let γ_n^- denote the part of the path γ_n between $\mathbb{V}_{\mathcal{G}_n}$ and \hat{v}_{n+M} . Let Γ be any path from u to \mathcal{V}_n disjoint from γ_n^- . Note that

$$T(u, \hat{v}_{n+M}) \leq (|\mathbb{E}_{\mathcal{G}_n}| + |\gamma_n^-|) t' \quad \text{and} \quad T(\Gamma) \geq |\gamma_n^-| t''.$$

(Here γ_n^- is identified with its set of edges.) By the choice of M ,

$$T(\Gamma) - T(u, \hat{v}_{n+M}) \geq (t'' - t') |\gamma_n^-| - |\mathbb{E}_{\mathcal{G}_n}| t' \geq (t'' - t') M - |\mathbb{E}_{\mathcal{G}_n}| t' > 0.$$

This proves that $T(u, \hat{v}_{n+M}) < T(u, w)$ for all $w \in \mathcal{V}_n \setminus \{\hat{v}_{n+M}\}$. The proof of the remaining inequality in (2.6) is similar. \square

Assume from now on that t' , t'' and M are chosen in accordance with Lemma 2.3. We will next introduce an auxiliary random variable Δ . Throughout this paper, Δ will denote any bounded integer-valued random variable independent of $\{\tau_e\}_{e \in \mathbb{E}}$. The auxiliary random variable is not necessary in order to derive the regenerative behaviour we do in this section. In fact, Δ is of no importance to most of our results in this paper. We will in Section 3 set $\Delta \equiv 0$. However, Δ will play a rôle in Section 4, where we need to be more careful to prove monotonicity of mean and variance. At this point we do not specify its distribution further, other than having bounded support.

Let $\rho_I := \max\{n \in \mathbb{Z} : \mathbb{V}_{\mathcal{G}_n} \cap I \neq \emptyset\}$ denote the furthest initially infected level. Define

$$n_k := \rho_I + \Delta + k(2M + 1), \quad \text{for } k \in \mathbb{Z},$$

and note that the sequence of events $\{A_{n_k}\}_{k \in \mathbb{Z}}$ is readily seen to be i.i.d. Let $\kappa = \min\{k \geq 0 : A_{n_k} \text{ occurs}\}$ and set $\rho_0 := n_\kappa + M$. Define further

$$\begin{aligned} \rho_k &:= M + \min\{n_m : n_m > \rho_{k-1} \text{ and } A_{n_m} \text{ occurs}\}, & \text{for } k \geq 1, \\ \rho_k &:= M + \max\{n_m : n_m + M < \rho_{k-1} \text{ and } A_{n_m} \text{ occurs}\}, & \text{for } k \leq -1. \end{aligned}$$

Since $\{A_{n_k}\}_{k \in \mathbb{Z}}$ is i.i.d. and $P(A_{n_k}) > 0$, the second Borel-Cantelli lemma gives that

$$P(A_{n_k} \text{ occurs for infinitely many } k \geq 0) = 1.$$

The same holds for $k \leq 0$. This generates a sequence $\{\rho_k\}_{k \in \mathbb{Z}}$, where $-\infty < \rho_k < \infty$ almost surely.

Note that $\rho_k \geq \rho_I + M$ for $k \geq 0$. Thus, for $k \geq 0$, Lemma 2.3 says that each path along which any vertex at level $\rho_k + M$ and beyond is infected has to pass the vertex \hat{v}_{ρ_k} .

Definition 2.4. A vertex \hat{v}_n will be referred to as a regeneration point if $n = \rho_k$ for some $k \geq 0$.

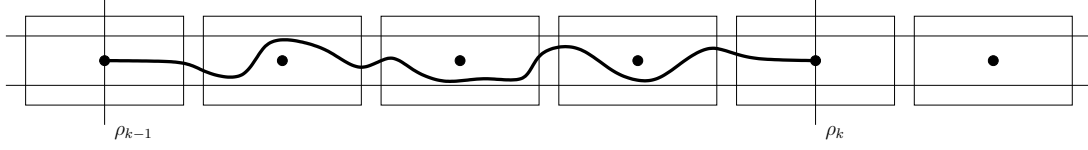


Figure 4: A schematic picture of a graph, in which boxes indicate locations of the sequence $\{A_{n_k}\}_{k \geq 0}$, vertical lines indicate the sequence $\{\rho_k\}_{k \geq 0}$, and dots indicate $\{\hat{v}_{n_k}\}_{k \geq 0}$. The distance between the two vertical lines is S_k , and the thick curve indicates τ_{S_k} .

For $k \in \mathbb{Z}$, define

$$S_k := \rho_k - \rho_{k-1}, \quad \text{and} \quad \tau_{S_k} := T(\hat{v}_{\rho_{k-1}}, \hat{v}_{\rho_k}).$$

For $k \geq 1$, S_k denotes the distance (measured in levels) between two regeneration points, and τ_{S_k} denotes the passage time between two regeneration points. By Lemma 2.3 we see that $\tau_{S_k} = T(\hat{v}_{\rho_k}) - T(\hat{v}_{\rho_{k-1}})$ for $k \geq 1$. With this notation, we may for $n \geq 0$ write the level of the n th regeneration, and the time it takes for the infection to reach the n th regeneration, as

$$\rho_n = \rho_0 + \sum_{k=1}^n S_k, \quad \text{and} \quad T(\hat{v}_{\rho_n}) = T(\hat{v}_{\rho_0}) + \sum_{k=1}^n \tau_{S_k},$$

respectively.

Lemma 2.5. *Assume that t' , t'' and M are chosen in accordance with Lemma 2.3. Then, $\{(\tau_{S_k}, S_k)\}_{k \in \mathbb{Z}}$ forms a sequence of i.i.d. $[0, \infty) \times \mathbb{Z}_+$ -valued random variables.*

Proof. That $\{S_k\}_{k \in \mathbb{Z}}$ is an i.i.d. sequence of geometrically distributed random variables, times a factor $2M + 1$ can easily be seen, since the events A_{n_k} are pairwise independent with equal success probabilities.

Note that τ_{S_k} is a random variable of the form $T(\hat{v}_{n_i+M}, \hat{v}_{n_j+M})$, for some $i < j$, conditioned (in particular) on the occurrence of the events A_{n_i} and A_{n_j} . Thus, independence of τ_{S_k} and τ_{S_l} for $k \neq l$ follows from Lemma 2.3 part b). That they are identically distributed is due to the events A_{n_k} being pairwise independent with equal success probabilities. \square

Proposition 2.6. *The sequence $\{T(\hat{v}_n)\}_{n \geq 1}$ is a regenerative sequence. Moreover, if t' , t'' and M are chosen in accordance with Lemma 2.3, then $\{\rho_n\}_{n \geq 0}$ is a sequence of regenerative levels for $\{T(\hat{v}_n)\}_{n \geq 1}$, such that*

$$T(v_{\rho_n+m,i}) - T(\hat{v}_{\rho_n}) \stackrel{d}{=} T(v_{\rho_1+m,i}) - T(\hat{v}_{\rho_1}), \quad \text{for all } m \geq M, n \geq 1,$$

where superscript d indicates that the equality holds in distribution.

Proof. That $\{T(\hat{v}_n)\}_{n \geq 1}$ is a regenerative sequence with sequence of regenerative levels $\{\rho_n\}_{n \geq 0}$ follows from Lemma 2.5. By Lemma 2.3, $T(v_{\rho_n+m,i}) - T(\hat{v}_{\rho_n}) = T(\hat{v}_{\rho_n}, v_{\rho_n+m,i})$ for $m \geq M$, whose distribution is independent of n , by definition of A_n . \square

Let $\mu_\tau := \mathbb{E}[\tau_{S_k}]$ and $\mu_S := \mathbb{E}[S_k]$ denote the expected passage time and distance between two regeneration points, respectively, and define

$$\mu := \frac{\mu_\tau}{\mu_S}, \quad \text{and} \quad \sigma^2 := \frac{\text{Var}(\tau_{S_k} - \mu S_k)}{\mu_S}. \quad (2.7)$$

It is immediate from the construction that the distributions of S_k and τ_{S_k} (and therefore also μ and σ^2) does not depend on the set of initially infected vertices I , nor on Δ . We will in next section see that μ and σ^2 appear as the constants that figure in Theorem 1.3, 1.4 and 1.5. In order to state clear moment conditions, we will also need to know how moments of τ_e relate to moments of S_k and τ_{S_k} . This is given in the following proposition.

Proposition 2.7. *Assume that the passage time distribution P_τ does not concentrate all mass in a single point. Then,*

a) *there exists an $\alpha > 0$ such that $\mathbb{E}[e^{\alpha S_k}] < \infty$.*

Assume further that there are $p \geq 1$ (edge) disjoint paths from \hat{v}_0 to \hat{v}_1 . Let $Y = \min(\tau_1, \dots, \tau_p)$, where τ_1, \dots, τ_p are independent and distributed as P_τ . Then,

b) *if $\mathbb{E}[Y^\alpha] < \infty$, for some $\alpha > 0$, we have $0 < \mathbb{E}[\tau_{S_k}^\alpha] < \infty$.*

In particular, if $\mathbb{E}[\tau_e^\alpha] < \infty$, then $\mathbb{E}[\tau_{S_k}^\alpha] < \infty$, and if $\mathbb{E}[\tau_{S_k}^\alpha] < \infty$ for $\alpha = 1$, and $\alpha = 2$ respectively, then

$$0 < \mu < \infty, \quad \text{and} \quad 0 < \sigma^2 < \infty.$$

Proof. a) Recall that if θ is geometrically distributed with parameter $p_A = P(A_n)$, then $S_k \stackrel{d}{=} (2M+1)\theta$. In particular, $0 < \mathbb{E}[S_k^\alpha] < \infty$ for $\alpha > 0$. Moreover,

$$\begin{aligned} \mathbb{E}[e^{\alpha S_k}] &= \sum_{n=1}^{\infty} e^{\alpha(2M+1)n} (1-p_A)^{n-1} p_A \\ &= e^{\alpha(2M+1)} p_A \sum_{n=1}^{\infty} \left(e^{\alpha(2M+1)} (1-p_A) \right)^{n-1} < \infty, \end{aligned}$$

if $e^{\alpha(2M+1)}(1-p_A) < 1$.

b) Let $\Gamma_j^{(1)}, \dots, \Gamma_j^{(p)}$ denote the p disjoint paths from \hat{v}_{j-1} to \hat{v}_j . Note that subadditivity gives

$$\tau_{S_k} \leq \sum_{j=\rho_{k-1}+1}^{\rho_k} T(\hat{v}_{j-1}, \hat{v}_j). \quad (2.8)$$

For any edge $e \in \mathbb{E}$ we have

$$P(\tau_e > t | A_n) \leq P(A_n)^{-1} P(\tau_e > t), \quad \text{and} \quad P(\tau_e > t | A_n^c) \leq P(A_n^c)^{-1} P(\tau_e > t). \quad (2.9)$$

Set $\Lambda_n := \{S_k = (2M+1)n\}$. Note that Λ_n is of the form $\bigcap_{i \in I} A_{n_i} \bigcap_{j \in J} A_{n_j}^c$ for disjoint sets $I, J \subseteq \{l, l+1, \dots, l+n\}$, where l is such that $n_l = \rho_{k-1} - M$. Hence, it follows from (2.9) that when $e \in \Gamma_j^{(i)}$ for some $i = 1, \dots, p$ and $j = \rho_{k-1} + 1, \dots, \rho_k$, then

$$P(\tau_e > t | \Lambda_n) \leq C_1 P(\tau_e > t), \quad (2.10)$$

where $C_1 = \max(P(A_n)^{-1}, P(A_n^c)^{-1})$. We will next prove that

$$\mathbb{E}[T(\hat{v}_{j-1}, \hat{v}_j)^\alpha | \Lambda_n] \leq C_2 \mathbb{E}[Y^\alpha], \quad (2.11)$$

for $j = \rho_{k-1} + 1, \dots, \rho_k$ and some $C_2 < \infty$. Let λ denote the length of the longest of the paths $\Gamma_j^{(i)}$. Then (2.11) follows immediately from

$$\begin{aligned} P(T(\hat{v}_{j-1}, \hat{v}_j)^\alpha > t | \Lambda_n) &\leq \prod_{i=1}^p P(T(\Gamma_j^{(i)}) > t^{1/\alpha} | \Lambda_n) \leq \prod_{i=1}^p \left(\sum_{e \in \Gamma_j^{(i)}} P(\tau_e > t^{1/\alpha}/\lambda | \Lambda_n) \right) \\ &\leq C_1^p \lambda^p P(\tau_e > t^{1/\alpha}/\lambda)^p = C_1^p \lambda^p P(Y^\alpha > t/\lambda^\alpha), \end{aligned}$$

where the second inequality follows since $T(\Gamma_j^{(i)}) \geq s$ implies that at least one of the edges $e \in \Gamma_j^{(i)}$ has $\tau_e > s/\lambda$, and the third inequality follows from (2.10). Combining (2.8) and (2.11) we deduce that

$$\begin{aligned} \mathbb{E}[\tau_{S_k}^\alpha] &\leq \sum_{n=1}^{\infty} \mathbb{E} \left[\left(\sum_{j=\rho_{k-1}+1}^{\rho_k} T(\hat{v}_{j-1}, \hat{v}_j) \right)^\alpha \middle| \Lambda_n \right] P(\Lambda_n) \\ &\leq \sum_{n=1}^{\infty} n^\alpha \sum_{j=\rho_{k-1}+1}^{\rho_{k-1}+n} \mathbb{E}[T(\hat{v}_{j-1}, \hat{v}_j)^\alpha | \Lambda_n] P(\Lambda_n) \\ &\leq C_2 \sum_{n=1}^{\infty} n^{\alpha+1} \mathbb{E}[Y^\alpha] P(\Lambda_n) \leq C_2 \mathbb{E}[Y^\alpha] \mathbb{E}[S_k^{\alpha+1}], \end{aligned} \quad (2.12)$$

where the second inequality follows since for any non-negative numbers a_j we have

$$\left(\sum_{j=1}^n a_j \right)^\alpha \leq (n \max_j a_j)^\alpha \leq n^\alpha \sum_{j=1}^n a_j^\alpha. \quad (2.13)$$

Thus, $\mathbb{E}[\tau_{S_k}^\alpha] < \infty$ from part a). We can conclude that $\mathbb{E}[\tau_{S_k}^\alpha] > 0$, since the passage times of all edges connecting level $\rho_{k-1} + M$ and $\rho_{k-1} + M + 1$ are independent of Λ_n . \square

Remark 2.8. It is worth pointing out that the initially infected component does not need to be finite. But, there needs to be a level m beyond which no vertex is initially infected. Proposition 2.6 holds also in this case. \square

3 Asymptotics for first-passage percolation

In this section we will present some variants of classical results for i.i.d. sequences, but here for first-passage percolation considered on essentially 1-dimensional periodic graphs. The fact that $\{T(\hat{v}_n)\}_{n \geq 1}$ is a regenerative sequence, and in particular, that $\{\tau_{S_k}\}_{k \geq 1}$ and $\{S_k\}_{k \geq 1}$ form i.i.d. sequences, will play a central rôle. We will assume throughout this section that t' , t'' and M

are chosen in accordance with Lemma 2.3, and that the auxiliary variable $\Delta \equiv 0$. In order to approximate $T(\hat{v}_n)$, we will stop the sequence $\{T(\hat{v}_{\rho_k})\}_{k \geq 0}$ in a suitable way. The asymptotic behaviour of stopped sums of this form, so called *stopped random walks*, i.e., random walks stopped by some stopping time, has been studied before. Gut (2009) treats this subject. Once the regenerative behaviour is known, results as Theorem 1.3 and 1.5 are easily obtained from the classical Law of large numbers and Law of the iterated logarithm. So, there is in these cases no need to refer to the theory for stopped sums. However, Theorem 1.4 and 3.6 would require more work, and we will base our proofs of these results on known results for stopped random walks. We should mention that apart from the results presented here, it is possible to deduce other results, such as stable laws, from known results for stopped random walks.

We will without further comment use the fact that if $Y_n \rightarrow Y$ and $\eta_n \rightarrow \infty$ almost surely as $n \rightarrow \infty$, then $Y_{\eta_n} \rightarrow Y$ almost surely as $n \rightarrow \infty$. We also remind the reader that for any i.i.d. sequence $\{Y_n\}_{n \geq 1}$, a simple application of the Borel-Cantelli lemmas shows that

$$\lim_{n \rightarrow \infty} \frac{Y_n^\alpha}{n} = 0, \text{ almost surely} \quad \Leftrightarrow \quad \mathbb{E}[|Y_1|^\alpha] < \infty. \quad (3.1)$$

To see this, note that $\mathbb{E}[|Y_1|^\alpha] < \infty$ is equivalent to

$$\sum_{n=1}^{\infty} P(|Y_n|^\alpha > \epsilon n) < \infty, \quad \text{for any } \epsilon > 0.$$

This is by the Borel-Cantelli lemmas equivalent to

$$\lim_{n \rightarrow \infty} \frac{|Y_n|^\alpha}{n} \leq \epsilon,$$

and (3.1) follows.

In order to approximate $T(\hat{v}_n)$, we will stop the regenerating sequence when A_{n_k} occurs for the least k such that $n_k \geq n$. In terms of the sequence of regenerative levels, we define

$$\nu(n) := \min\{m \geq 0 : \rho_m \geq n + M\}.$$

Lemma 3.1. $\{\nu(n)\}_{n \geq 0}$ is a non-decreasing sequence such that

- a) $\lim_{n \rightarrow \infty} \frac{n}{\nu(n)} = \mu_S, \quad \text{almost surely.}$
- b) $\lim_{n \rightarrow \infty} \frac{\rho_{\nu(n)}}{n} = 1, \quad \text{almost surely.}$

Proof. It is clear that $\nu(n) \uparrow \infty$ as $n \rightarrow \infty$. Lemma 2.5 and Proposition 2.7 assure that $\{S_k\}_{k \geq 1}$ forms an i.i.d. sequence with finite mean. Since $\rho_{\nu(n)-1} < n + M \leq \rho_{\nu(n)}$, we have

$$\frac{\rho_{\nu(n)}}{\nu(n)} - \frac{S_{\nu(n)}}{\nu(n)} < \frac{n + M}{\nu(n)} \leq \frac{\rho_{\nu(n)}}{\nu(n)}.$$

This, together with the classical Law of large numbers and (3.1) proves a). Since

$$\frac{\rho_{\nu(n)}}{n} = \frac{\rho_{\nu(n)}}{\nu(n)} \frac{\nu(n)}{n},$$

part b) follows from the Law of large numbers and part a). □

Recall that $\{\hat{T}_n\}_{n \geq 1}$ denotes either of $\{T_n\}_{n \geq 1}$, $\{\max_{v \in \mathbb{V}_{\mathcal{G}_n}} T(v)\}_{n \geq 1}$ and $\{T(v_n)\}_{n \geq 1}$, where $\{v_n\}_{n \geq 1}$ is any sequence of vertices such that v_n is at level n . When we prove Theorem 1.3, 1.4 and 1.5, we will first obtain the results for the stopped sequence $\{T(\hat{v}_{\rho_{\nu(n)}})\}_{n \geq 1}$. What we then need to finish the proofs is summarized in the following lemma.

Lemma 3.2. *Assume that there are $p \geq 1$ (edge) disjoint paths from \hat{v}_0 to \hat{v}_1 . Let $Y = \min(\tau_1, \dots, \tau_p)$, where τ_1, \dots, τ_p are independent and distributed as P_τ . Then, for any $\alpha > 0$,*

- a) $\lim_{n \rightarrow \infty} \frac{|\rho_{\nu(n)} - n|^\alpha}{n} = 0$, almost surely.
- b) if $E[Y^\alpha] < \infty$, then $\lim_{n \rightarrow \infty} \frac{|T(\hat{v}_n) - T(\hat{v}_{\rho_{\nu(n)}})|^\alpha}{n} = 0$, almost surely.
- c) if $E[\tau_e^\alpha] < \infty$, then $\lim_{n \rightarrow \infty} \frac{|\hat{T}_n - T(\hat{v}_n)|^\alpha}{n} = 0$, almost surely.

Proof. Since $\rho_{\nu(n)} - n \leq S_{\nu(n)} + M \stackrel{d}{=} S_k + M$, then a) follows from (3.1) and part a) of Proposition 2.7. By subadditivity

$$\left| T(\hat{v}_{\rho_{\nu(n)}}) - T(\hat{v}_n) \right| \leq \sum_{j=n+1}^{\rho_{\nu(n)}} T(\hat{v}_{j-1}, \hat{v}_j) \leq \sum_{j=\rho_{\nu(n)}-1-M+1}^{\rho_{\nu(n)}} T(\hat{v}_{j-1}, \hat{v}_j),$$

which in the proof of Proposition 2.7 was seen to have finite moment of the same order as Y . Thus, also b) follows from (3.1). Finally, also c) follows from (3.1). Note that

$$T_n \leq T(v_n) \leq \max_{v \in \mathbb{V}_{\mathcal{G}_n}} T(v) \leq T_n + \sum_{e \in \mathbb{E}_{\mathcal{G}_n}} \tau_e,$$

implies that $|\hat{T}_n - T(\hat{v}_n)| \leq \sum_{e \in \mathbb{E}_{\mathcal{G}_n}} \tau_e$, which via (2.13) is easily seen to have finite moment of same order as τ_e . \square

3.1 Proof of point-wise limit theorems

Proof of Theorem 1.3. Almost sure convergence. Lemma 2.5 and Proposition 2.7 gives that $\{\tau_{S_k}\}_{k \geq 1}$ is an i.i.d. sequence with finite mean. Thus, as $n \rightarrow \infty$,

$$\frac{T(\hat{v}_{\rho_{\nu(n)}})}{n} = \frac{T(\hat{v}_{\rho_0}) + \sum_{k=1}^{\nu(n)} \tau_{S_k}}{\nu(n)} \frac{\nu(n)}{n} \rightarrow \frac{\mu_\tau}{\mu_S}, \quad \text{almost surely,}$$

according to the classical Law of large numbers and Lemma 3.1. We conclude that, as $n \rightarrow \infty$,

$$\frac{T(\hat{v}_n)}{n} = \frac{T(\hat{v}_{\rho_{\nu(n)}})}{n} + \frac{T(\hat{v}_n) - T(\hat{v}_{\rho_{\nu(n)}})}{n} \rightarrow \frac{\mu_\tau}{\mu_S}, \quad \text{almost surely,}$$

by part b) of Lemma 3.2. The almost sure convergence of \hat{T}_n/n now follows from part c) of the same lemma.

Uniform integrability. It suffices to prove that $\sup_{n \geq 1} \mathbb{E}[(\hat{T}_n/n)^r] < \infty$. According to subadditivity and (2.13)

$$\hat{T}_n^r \leq 3^r \left(T(\hat{v}_0)^r + \left(\sum_{j=1}^n T(\hat{v}_{j-1}, \hat{v}_j) \right)^r + \left(\sum_{e \in \mathbb{E}_{\mathcal{G}_n}} \tau_e \right)^r \right).$$

By convexity of the function x^r , we have

$$\left(\frac{1}{n} \sum_{k=1}^n T(\hat{v}_{j-1}, \hat{v}_j) \right)^r \leq \frac{1}{n} \sum_{k=1}^n T(\hat{v}_{j-1}, \hat{v}_j)^r.$$

Hence, to prove uniform integrability of $\{(\hat{T}_n/n)^r\}_{n \geq 1}$, we only have to conclude that $\mathbb{E}[T(\hat{v}_0)^r]$, $\mathbb{E}[T(\hat{v}_{j-1}, \hat{v}_j)^r]$ and $\mathbb{E}[(\sum_{e \in \mathbb{E}_{\mathcal{G}_n}} \tau_e)^r]$ are finite when $\mathbb{E}[\tau_e^r]$ is. This is easily seen via (2.13), since both $T(\hat{v}_0)$ and $T(\hat{v}_{j-1}, \hat{v}_j)$ can be bounded by the passage time of a finite number of edges. (Note that the regenerative behaviour was not used.)

L^r -convergence. The L^r -convergence now follows from the almost sure convergence and uniform integrability. \square

Theorem 1.4 will be deduced from the following result sometimes referred to as *Anscombe's theorem*. For a proof, we refer the reader to e.g. Gut (2009, Theorem 1.3.1).

Theorem 3.3 (Anscombe's theorem). *Let $\{\xi_k\}_{k \geq 1}$ be an i.i.d. sequence with mean zero and variance σ_ξ^2 . Assume further that*

$$\frac{\eta(n)}{n} \xrightarrow{p} \theta, \quad \text{in probability,}$$

as $n \rightarrow \infty$. Then, as $n \rightarrow \infty$,

$$\frac{\sum_{k=1}^{\eta(n)} \xi_k}{\sigma_\xi \sqrt{\theta n}} \xrightarrow{d} \chi, \quad \text{in distribution,}$$

where χ has a standard normal distribution.

Proof of Theorem 1.4. It follows from Lemma 2.5 and Proposition 2.7 that $\{\tau_{S_k} - \mu S_k\}_{k \geq 1}$ is an i.i.d. sequence with zero mean and finite variance. An application of Anscombe's theorem, together with Lemma 3.1, gives convergence in distribution of the former term in the right-hand side of

$$\frac{T(\hat{v}_n) - \mu n}{\sigma \sqrt{n}} = \frac{T(\hat{v}_{\rho_{\nu(n)}}) - \mu \rho_{\nu(n)}}{\sigma \sqrt{n}} + \frac{T(\hat{v}_n) - T(\hat{v}_{\rho_{\nu(n)}}) - \mu(n - \rho_{\nu(n)})}{\sigma \sqrt{n}},$$

to a standard normal distribution, as $n \rightarrow \infty$. The latter term in the above right-hand side vanishes according to part a) and b) of Lemma 3.2. The convergence of \hat{T}_n now follows from part c) of the same lemma. \square

Theorem 1.5 will be proved from a version of the Law of the iterated logarithm for i.i.d. sequences that is more general than the classical one. The classical version would suffice to prove the second statement in the theorem. A proof of the more general version for i.i.d. sequences can be found in e.g. Gut (2005).

Proof of Theorem 1.5. Recall that $\tau_{S_k} - \mu S_k$ are i.i.d. for $k \geq 1$, with zero mean and finite variance, due to Lemma 2.5 and Proposition 2.7. Trivially

$$\begin{aligned} \frac{T(\hat{v}_n) - \mu n}{\sigma \sqrt{2n \log \log n}} &= \frac{T(\hat{v}_{\rho_{\nu(n)}}) - \mu \rho_{\nu(n)}}{\sigma \sqrt{\mu_S 2\nu(n) \log \log \nu(n)}} \sqrt{\mu_S \frac{\nu(n)}{n}} \sqrt{\frac{\log \log \nu(n)}{\log \log n}} \\ &+ \frac{T(\hat{v}_n) - T(\hat{v}_{\rho_{\nu(n)}}) - \mu(n - \rho_{\nu(n)})}{\sigma \sqrt{2n \log \log n}}. \end{aligned} \quad (3.2)$$

Since $\nu(n)$ is non-decreasing, and for each $m \in \mathbb{Z}_+$, there is an $n \in \mathbb{Z}_+$ such that $\nu(n) = m$, it follows from the extended version of the Law of the iterated logarithm for i.i.d. sequences that

$$\mathcal{L} \left(\left\{ \frac{T(\hat{v}_{\rho_{\nu(n)}}) - \mu \rho_{\nu(n)}}{\sigma \sqrt{\mu_S 2\nu(n) \log \log \nu(n)}} \right\}_{n: \nu(n) \geq 3} \right) = [-1, 1], \quad \text{almost surely,}$$

and, in particular, that almost surely

$$\limsup_{n \rightarrow \infty} \frac{T(\hat{v}_{\rho_{\nu(n)}}) - \mu \rho_{\nu(n)}}{\sigma \sqrt{\mu_S 2\nu \log \log \nu}} = 1, \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{T(\hat{v}_{\rho_{\nu(n)}}) - \mu \rho_{\nu(n)}}{\sigma \sqrt{\mu_S 2\nu \log \log \nu}} = -1.$$

Lemma 3.1 gives that $\mu_S \nu(n)/n \rightarrow 1$, almost surely, as $n \rightarrow \infty$, and we further conclude that

$$\lim_{n \rightarrow \infty} \frac{\log \log \nu(n)}{\log \log n} = \lim_{n \rightarrow \infty} \frac{\log \left(\log n + \log \frac{\nu(n)}{n} \right)}{\log \log n} = 1, \quad \text{almost surely.}$$

An application of Lemma 3.2 now completes the proof. \square

Remark 3.4. Say that we are interested in the sequence $\{T(\hat{v}_0, \hat{v}_n)\}_{n \geq 1}$. A closer look at the proofs of Theorem 1.3, 1.4 and 1.5 reveals that the only moment condition required in order to prove the various modes of convergence for $\{T(\hat{v}_0, \hat{v}_n)\}_{n \geq 1}$ is not $E[\tau_e^\alpha] < \infty$ for given values of α , but $E[\tau_{S_k}^\alpha] < \infty$, for $k \geq 1$ and corresponding α . According to Proposition 2.7, this holds when $E[Y^\alpha] < \infty$, where Y denotes the minimum of p independent passage times, and p is the number of disjoint paths from \hat{v}_0 to \hat{v}_1 . As an example, on any (K, d) -tube with $K, d \geq 2$, the passage time distribution given by

$$P(\tau_e > x) = x^{-\alpha}, \quad \text{for } x > 1, \quad (3.3)$$

for some $\alpha > 0$, satisfies $E[\tau_e^\alpha] = \infty$ but $E[\min(\tau_1, \tau_2)^\alpha] < \infty$. Hence, Theorem 1.3, 1.4 and 1.5 holds in this example for the sequence $\{T(\hat{v}_0, \hat{v}_n)\}_{n \geq 1}$ and corresponding values of α , even though $E[\tau_e^\alpha] = \infty$. \square

3.2 One dimensional shape theorem

Theorem 1.5 gives the precise rate of convergence towards the asymptotic shape B^* in the case of (K, d) -tubes. We will next rephrase this in terms of the set of infected vertices. The following corollary gives the precise rate of fluctuations of that set. Recall that $B^* = B^*(t) = [-\mu_K^{-1}, \mu_K^{-1}] \times [0, K/t]^{d-1}$.

Corollary 3.5. *Consider first-passage percolation on a (K, d) -tube with $E[\tau_e^2] < \infty$. We have for all $\lambda > \sigma\sqrt{2/\mu_K}$, almost surely, that*

$$\left(1 - \lambda\sqrt{t^{-1}\log\log t}\right) B^* \subset \frac{1}{t}\tilde{B}_t \subset \left(1 + \lambda\sqrt{t^{-1}\log\log t}\right) B^*, \quad (3.4)$$

for all t large enough. Moreover, for all $\lambda < \sigma\sqrt{2/\mu_K}$ and $s \geq 0$, for either of the inclusions in (3.4), there exists, almost surely, $t \geq s$ such that the inclusion does not hold.

Proof. Fix $\epsilon > 0$. By Theorem 1.5, there exists an almost surely finite $N = N(\epsilon)$ such that

$$\mu_K n - (1 + \epsilon)\sigma\sqrt{2n\log\log n} < \min(T_{-n}, T_n),$$

for all $n \geq N$. This implies that

$$\tilde{B}_t \subseteq [-n, n] \times [0, K]^{d-1},$$

for all

$$t \leq \mu_K n - (1 + \epsilon)\sigma\sqrt{2n\log\log n} \quad (3.5)$$

and $n \geq N$. Write n_t for the least n such that (3.5) holds. By the choice of n_t ,

$$\frac{t}{\mu_K} \geq n_t - 1 - \frac{1 + \epsilon}{\mu_K}\sigma\sqrt{2(n_t - 1)\log\log(n_t - 1)} = (n_t - 1)g(n_t),$$

for some increasing function g such that $g(n) \rightarrow 1$ as $n \rightarrow \infty$. It follows that

$$n_t - 1 \leq \frac{1}{\mu_K} \left(t + (1 + \epsilon)\sigma\sqrt{\frac{2t}{\mu_K g(n_t)} \log\log \frac{t}{\mu_K g(n_t)}} \right)$$

Since $n_t \rightarrow \infty$, also $g(n_t) \rightarrow 1$, as $t \rightarrow \infty$. Since $\epsilon > 0$ was arbitrary, we have shown that for all $\epsilon > 0$ there exists an almost surely finite $T = T(\epsilon)$ such that

$$\tilde{B}_t \subseteq \left(t + (1 + \epsilon)\sigma\sqrt{\frac{2t}{\mu_K} \log\log t} \right) B^*$$

for all $t \geq T$. The proof of the lower inclusion in (3.4) follows in a similar way from Theorem 1.5 applied to $\max_{v \in \mathbb{V}_{\mathcal{G}_n}} T(v)$.

It remains to prove the second statement of the corollary. Fix $\epsilon > 0$. It follows from Theorem 1.5 that for all $n \geq 1$ there exists, almost surely, $N = N(\epsilon) \geq n$ such that

$$T_N \leq \mu_K N - (1 - \epsilon)\sigma\sqrt{2N\log\log N}$$

In particular,

$$\tilde{B}_{t_N} \not\subseteq [-N, N] \times [0, K]^{d-1}$$

for $t_N := \mu_K N - (1 - \epsilon)\sigma\sqrt{2N\log\log N}$. Since $t_N \leq \mu_K N$, it follows that

$$\tilde{B}_{t_N} \not\subseteq \left(t_N + (1 - \epsilon)\sigma\sqrt{\frac{2t_N}{\mu_K} \log\log \frac{t_N}{\mu_K}} \right) B^*.$$

Since $\epsilon > 0$ was arbitrary, we have shown that for any $\lambda < \sigma\sqrt{2/\mu_K}$, $\epsilon > 0$ and $s > 0$, there exists, almost surely, $t = t(\epsilon) \geq s$ such that the upper inclusion in (3.4) cannot hold. The failure of the lower inclusion follows in a similar way. \square

3.3 Functional Donsker theorem

Donsker's theorem can be seen as a functional version of the central limit theorem. In contrast to the central limit theorem that treats weak convergence of real-valued random variables, Donsker's theorem treats weak convergence of sequences of real-valued random functions. Let $D = D[0, \infty)$ denote the set of right-continuous functions with left-hand limits on $[0, \infty)$. Let \mathcal{D} denote the σ -algebra generated by the open sets in D with Skorohod's J_1 -topology, defined by the following. Let Λ denote the set of strictly increasing, continuous mappings of $[0, b]$ onto itself. A sequence $\{f_n\}_{n \geq 1}$ of elements in D is said to be J_1 -convergent to f if, for every $b \geq 0$, there exists a sequence $\{\lambda_n\}_{n \geq 1}$ in Λ such that

$$\sup_{0 \leq t \leq b} |\lambda_n(t) - t| \rightarrow 0, \quad \text{and} \quad \sup_{0 \leq t \leq b} |f_n(\lambda_n(t)) - f(t)| \rightarrow 0,$$

as $n \rightarrow \infty$. If $\{P_n\}_{n \geq 1}$ is a sequence of probability measures on the measurable space (D, \mathcal{D}) , then we say that P_n converge weakly to P , denoted $P_n \xrightarrow{J_1} P$, if

$$\int_D f dP_n \rightarrow \int_D f dP,$$

for all bounded, continuous f from D to \mathbb{R} .

Let $\{\xi_k\}_{k \geq 1}$ be an i.i.d. sequence of random variables with zero mean and variance $\sigma_\xi^2 < \infty$, set $S_n = \sum_{k=1}^n \xi_k$, and define

$$X_n(t) := \frac{1}{\sigma_\xi \sqrt{n}} S_{[nt]}, \quad \text{for } t \geq 0.$$

Donsker's theorem states that $X_n \xrightarrow{J_1} W$, as $n \rightarrow \infty$, where W denotes Wiener measure. The following is a result in the same spirit, for our first-passage percolation process.

Theorem 3.6 (Functional Donsker theorem). *If $E[\tau_e^2] < \infty$, then*

$$\frac{\hat{T}_{[nt]} - \mu[nt]}{\sigma \sqrt{n}} \xrightarrow{J_1} W, \quad \text{as } n \rightarrow \infty.$$

As for the point-wise central limit theorem, there is an Anscombe version of Donsker's theorem. We will use it as a lemma to prove Theorem 3.6. Suppose that $\{\eta(n)\}_{n \geq 0}$ is a non-decreasing, right-continuous family of positive, integer valued random variables such that $\eta(n)/n \rightarrow \theta$, almost surely, as $n \rightarrow \infty$. Define

$$Y_n(t) := \frac{1}{\sigma_\xi \sqrt{n}} S_{\eta([nt])}, \quad \text{for } t \geq 0.$$

An Anscombe-version of Donsker's theorem states the following.

Lemma 3.7.

$$\theta^{-1/2} Y_n \xrightarrow{J_1} W, \quad \text{as } n \rightarrow \infty.$$

We refer the reader to Gut (2009, Theorem 5.2.1) for a proof of Lemma 3.7. The lemma is of great interest in its own right, but we restate it here as a lemma in order to maintain focus on our main aim. We will deduce Theorem 3.6 from Lemma 3.7.

Proof of Theorem 3.6. Lemma 2.5 and Proposition 2.7 assure that $\{\tau_{S_k} - \mu S_k\}_{k \geq 1}$ is an i.i.d. sequence with zero mean and finite variance. From Lemma 3.7 it follows that

$$\frac{T(\hat{v}_{\rho_{\nu}(\lfloor nt \rfloor)}) - \mu \rho_{\nu}(\lfloor nt \rfloor)}{\sigma \sqrt{n}} \xrightarrow{J_1} W, \quad \text{as } n \rightarrow \infty.$$

It remains to prove that, as $n \rightarrow \infty$,

$$\sup_{0 \leq t \leq b} \left| \frac{\hat{T}_{\lfloor nt \rfloor} - T(\hat{v}_{\rho_{\nu}(\lfloor nt \rfloor)}) - \mu(\lfloor nt \rfloor - \rho_{\nu}(\lfloor nt \rfloor))}{\sigma \sqrt{n}} \right| \rightarrow 0, \quad \text{almost surely.} \quad (3.6)$$

According to Lemma 3.2, as $n \rightarrow \infty$,

$$\left| \frac{\hat{T}_n - T(\hat{v}_{\rho_{\nu}(n)}) - \mu(n - \rho_{\nu}(n))}{\sigma \sqrt{n}} \right| \rightarrow 0, \quad \text{almost surely.} \quad (3.7)$$

For any sequence of real numbers $\{x_n\}_{n \geq 1}$ it holds that

$$\lim_{n \rightarrow \infty} \frac{x_n}{\sqrt{n}} = 0 \quad \Rightarrow \quad \lim_{n \rightarrow \infty} \frac{\max_{k \leq bn} |x_k|}{\sqrt{n}} = 0. \quad (3.8)$$

To see this, fix $\epsilon > 0$, and choose N such that $|x_n|/\sqrt{n} \leq \epsilon$ for all $n \geq N$. Then the left-hand side in (3.8) can be made arbitrarily small for large n , since

$$\frac{\max_{k \leq bn} |x_k|}{\sqrt{n}} = \frac{\max_{k < N} |x_k|}{\sqrt{n}} + \frac{\max_{N \leq k \leq bn} |x_k|}{\sqrt{n}} \leq \epsilon + \epsilon \sqrt{b},$$

if n is chosen large enough.

In fact, (3.8) improves (3.7), and we get

$$\frac{\max_{k \leq bn} \left| \hat{T}_k - T(\hat{v}_{\rho_{\nu}(k)}) - \mu(k - \rho_{\nu}(k)) \right|}{\sigma \sqrt{n}} \rightarrow 0, \quad \text{almost surely,}$$

as $n \rightarrow \infty$. But this is equivalent to (3.6). □

Remark 3.8. Remark 3.4 applies also to Theorem 3.6. □

4 Monotonicity of mean and variance

In this section we will prove Theorem 1.6 and 1.7. This shows that mean and variance of $T(v_{n,i})$ are monotonous in n , if n is sufficiently large. The method of proof will use the regenerative behaviour explored in Section 2. In this section, the auxiliary random variable Δ introduced in

Section 2 will turn out useful. Throughout this section, Δ will be assumed uniformly distributed on $\{0, 1, \dots, 2M\}$.

From what is known for stopped random walks (see e.g. Gut (2009, Theorem 4.2.4)), it follows that $E[T(\hat{v}_{\rho_{\nu(n)}})] = \mu n + C$, for some constant C , and $\text{Var}(T(\hat{v}_{\rho_{\nu(n)}})) = \sigma^2 n + o(n)$, for large n . We will in this section need an essential amount of extra work, in order to improve the latter statement and prove that there is a constant C , such that $\text{Var}(T(\hat{v}_{\rho_{\nu(n)}})) = \sigma^2 n + C$. What then remains in order to prove Theorem 1.6 and 1.7, is to show that the differences between $E[T(v_{n,i})]$ and $E[T(\hat{v}_{\rho_{\nu(n)}})]$, and between $\text{Var}(T(v_{n,i}))$ and $\text{Var}(T(\hat{v}_{\rho_{\nu(n)}}))$, converge as $n \rightarrow \infty$. We will present full proofs of Theorem 1.6 and 1.7 based on Wald's lemma.

Lemma 4.1 (Wald's lemma). *Let ξ_1, ξ_2, \dots be i.i.d. random variables with mean μ_ξ , and set $S_n = \sum_{k=1}^n \xi_k$. Let N be a stopping time with $E[N] < \infty$.*

- a) $E[S_N] = \mu_\xi E[N]$.
- b) If $\sigma_\xi^2 = \text{Var}(\xi_1) < \infty$, then $E[(S_N - \mu_\xi N)^2] = \sigma_\xi^2 E[N]$.
- c) If X is independent of ξ_1, ξ_2, \dots , then $E[XS_N] = \mu_\xi E[XN]$. In particular, $\text{Cov}(X, S_N) = \mu_\xi \text{Cov}(X, N)$.

A proof of Wald's lemma can be found e.g. in Gut (2009, Theorem 1.5.3). The third part of the lemma is a slight extension of the first part, and proved in an analogous way. If $\mathcal{F}_n = \sigma(\{(\rho_0, T(\hat{v}_{\rho_0})), (S_1, \tau_{S_1}), \dots, (S_n, \tau_{S_n})\})$, then it is immediate from the definition that $\nu(n)$ is a stopping time with respect to the sequence of σ -algebras $\{\mathcal{F}_n\}_{n \geq 1}$.

The importance of the auxiliary variable Δ is the following. A regeneration point may only occur every $2M + 1$ levels. However, introducing a shift uniformly distributed on $\{0, 1, \dots, 2M\}$ allows every level equal probability to be included in the subset of levels at which regeneration points may occur. This is precisely the rôle of Δ . This allows the following lemma, as well as details in the proof of Theorem 1.6 and 1.7, to become less messy.

Lemma 4.2. *For $n \geq \rho_I$,*

$$E[\nu(n)] = \frac{n - \rho_I}{\mu_S}.$$

Proof. Assume that $n \geq \rho_I$. We may interpret $\nu(n)$ as the number of regeneration points before (but not including) level $n + M$. That is, the number of $k \geq 0$ such that A_{n_k} occurs for $n_k < n$. Since $n_0 = \rho_I + \Delta$, this number is at most $\left\lfloor \frac{n + 2M - \rho_I - \Delta}{2M + 1} \right\rfloor$. Since the shift Δ is independent of $\{\tau_e\}_{e \in \mathbb{E}}$, we can, conditioned on Δ , think of $\nu(n)$ as the number of successes in $\left\lfloor \frac{n + 2M - \rho_I - \Delta}{2M + 1} \right\rfloor$ independent Bernoulli trials, each with success probability $p_A := P(A_{n_k})$. Conditioning on Δ , we see that

$$E[\nu(n)] = p_A E \left[\left\lfloor \frac{n + 2M - \rho_I - \Delta}{2M + 1} \right\rfloor \right]. \quad (4.1)$$

If $n - \rho_I = (2M + 1)k$, for some $k \geq 0$, one realise from (4.1) that

$$E[\nu(n)] = \frac{p_A}{2M + 1} (2M + 1)k = \frac{n - \rho_I}{\mu_S},$$

where the latter equality follows from the fact that S_k is geometrically distributed with parameter p_A , times a factor $2M + 1$, that is, $\mu_S = (2M + 1)/p_A$. Again from (4.1), one realise that as $n - \rho_I$ increase from $(2M + 1)k$ to $(2M + 1)k + 2M$, then $E[\nu(n)]$ will have to increase with $p_A/(2M + 1)$ for each step. \square

We are now ready to prove Theorem 1.6 and 1.7.

Proof of Theorem 1.6. Wald's lemma together with Lemma 4.2 gives that

$$E \left[T(\hat{v}_{\rho_{\nu(n)}}) \right] = E \left[T(\hat{v}_{\rho_0}) + \sum_{k=1}^{\nu(n)} \tau_{S_k} \right] = E \left[T(\hat{v}_{\rho_0}) \right] + \mu_\tau E[\nu(n)] = E \left[T(\hat{v}_{\rho_0}) \right] + \mu n - \mu \rho_I.$$

It remains to prove that there is a finite constant C_i such that

$$E \left[T(v_{n,i}) - T(\hat{v}_{\rho_{\nu(n)}}) \right] \rightarrow C_i, \quad \text{as } n \rightarrow \infty. \quad (4.2)$$

Arguments of the type we will use to prove (4.2) will be used repeatedly in the proof of Theorem 1.7. For this reason, we present the argument in detail here. To make the argument clear, we will define a random variable to which $T(v_{n,i}) - T(\hat{v}_{\rho_{\nu(n)}})$ converges in distribution. The limit C_i will then be the expectation of this random variable.

Recall that $n_k = \rho_I + \Delta + k(2M + 1)$, set $m_{n,k} := n - (2M + 1)k$ for $k \geq 1$, and set

$$\begin{aligned} r_+ &:= M + \min\{n_k \geq 0 : A_{n_k} \text{ occurs}\}, \\ r_0 &:= M + \max\{m_{0,k} < 0 : A_{m_{0,k}} \text{ occurs}\}. \end{aligned}$$

Observe that r_+ denotes the first element of the sequence $\{\rho_k\}_{k \geq 0}$ greater than zero, whereas r_0 is not defined along the same subsequence of the integers as $\{\rho_k\}_{k \geq 0}$. Define

$$Y_{k,i} := T(\hat{v}_{r_0}, v_{k,i}), \quad \text{and} \quad Y_+ := T(\hat{v}_{r_0}, \hat{v}_{r_+}),$$

and the events

$$\begin{aligned} D_{T,n} &:= \{A_{m_{n,k}} \text{ occurs for some } k \text{ such that } \rho_I \leq m_{n,k} < n\}, \\ D_{Y,n} &:= \{A_{m_{0,k}} \text{ occurs for some } k \text{ such that } \rho_I \leq m_{0,k} + n < n\}. \end{aligned}$$

Clearly $P(D_{T,n}) = P(D_{Y,n}) \rightarrow 1$ as $n \rightarrow \infty$. Moreover,

$$\{T(v_{n,i}) - T(\hat{v}_{\rho_{\nu(n)}}) \leq t\} \cap D_{T,n} \stackrel{d}{=} \{Y_{0,i} - Y_+ \leq t\} \cap D_{Y,n}.$$

So, if we let $H_{T,n} = \{T(v_{n,i}) - T(\hat{v}_{\rho_{\nu(n)}}) \leq t\}$ and $H_Y = \{Y_{0,i} - Y_+ \leq t\}$, then as $n \rightarrow \infty$,

$$\begin{aligned} P(H_{T,n}) &= P(H_{T,n} \cap D_{T,n}) + P(H_{T,n} \cap D_{T,n}^c) \\ &= P(H_Y) + P(H_{T,n} \cap D_{T,n}^c) - P(H_Y \cap D_{Y,n}^c) \rightarrow P(H_Y). \end{aligned} \quad (4.3)$$

Thus, $T(v_{n,i}) - T(\hat{v}_{\rho_{\nu(n)}}) \xrightarrow{d} Y_0 - Y_+$ as $n \rightarrow \infty$. If, in addition, $\{T(v_{n,i}) - T(\hat{v}_{\rho_{\nu(n)}})\}_{n \geq 1}$ is uniformly integrable, then

$$E \left[T(v_{n,i}) - T(\hat{v}_{\rho_{\nu(n)}}) \right] \rightarrow E[Y_{0,i} - Y_+], \quad \text{as } n \rightarrow \infty.$$

To deduce uniform integrability, it suffices to prove that $\sup_{n \geq 1} \mathbb{E} [T(v_{n,i}) - T(\hat{v}_{\rho_{\nu(n)}})] < \infty$. Subadditivity gives that

$$T(v_{n,i}) - T(\hat{v}_{\rho_{\nu(n)}}) \leq T(v_{n,i}, \hat{v}_n) + \sum_{j=n+1}^{\rho_{\nu(n)}} T(\hat{v}_{j-1}, \hat{v}_j). \quad (4.4)$$

But, the distribution of the right-hand side of (4.4) does not depend on n . Thus, it suffices to see that it has finite expectation. Conditioning on $\Lambda_k = \{\rho_{\nu(n)} - n = k\}$, one may do so in an analogous way as in (2.12) in the proof of Proposition 2.7, part *b*). We omit the details. \square

The proof of Theorem 1.7 needs a little more work, due to arising covariance terms. Moment convergence arguments similar to the one made to prove (4.2) will be used repeatedly.

Proof of Theorem 1.7. To begin with,

$$\begin{aligned} \text{Var} (T(v_{n+2M,i})) &= \text{Var} (T(\hat{v}_{\rho_{\nu(n)}}) - \mu \rho_{\nu(n)}) \\ &\quad + \text{Var} (T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}}) + \mu \rho_{\nu(n)}) \\ &\quad + 2 \text{Cov} (T(\hat{v}_{\rho_{\nu(n)}}) - \mu \rho_{\nu(n)}, T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}})) \\ &\quad + 2\mu \text{Cov} (T(\hat{v}_{\rho_{\nu(n)}}), \rho_{\nu(n)}) - 2\mu^2 \text{Var} (\rho_{\nu(n)}). \end{aligned} \quad (4.5)$$

We will have to treat each of the terms on the right-hand side one by one. Consider the first term in the right-hand side in 4.5, and note that

$$\begin{aligned} \text{Var} (T(\hat{v}_{\rho_{\nu(n)}}) - \mu \rho_{\nu(n)}) &= \text{Var} \left(T(\hat{v}_{\rho_0}) - \mu \rho_0 + \sum_{k=1}^{\nu(n)} (\tau_{S_k} - \mu S_k) \right) \\ &= \mathbb{E} \left[\left(\sum_{k=1}^{\nu(n)} (\tau_{S_k} - \mu S_k) \right)^2 \right] + \text{Var} (T(\hat{v}_{\rho_0}) - \mu \rho_0) \\ &\quad + 2 \text{Cov} \left(\sum_{k=1}^{\nu(n)} (\tau_{S_k} - \mu S_k), T(\hat{v}_{\rho_0}) - \mu \rho_0 \right) \end{aligned}$$

So, an application of both the second and third part of Wald's lemma, together with Lemma 4.2, yield

$$\begin{aligned} \text{Var} (T(\hat{v}_{\rho_{\nu(n)}}) - \mu \rho_{\nu(n)}) &= \text{Var} (\tau_{S_k} - \mu S_k) \mathbb{E}[\nu(n)] + \text{Var} (T(\hat{v}_{\rho_0}) - \mu \rho_0) \\ &\quad + \mathbb{E}[\tau_{S_k} - \mu S_k] \text{Cov} (\nu(n), T(\hat{v}_{\rho_0}) - \mu \rho_0) \\ &= \sigma^2(n - \rho_I) + \text{Var} (T(\hat{v}_{\rho_0}) - \mu \rho_0). \end{aligned}$$

To conclude that $\text{Var}(\rho_{\nu(n)})$ is constant, interpret $\rho_{\nu(n)}$ as the level of the first regeneration after level n . Since a regeneration is equally likely to occur at any level, due to the shift variable Δ , it follows that $\text{Var}(\rho_{\nu(n)}) = \text{Var}(\rho_{\nu(n)} - n)$ is independent of n , and therefore constant.

All remaining terms in the right-hand side of (4.5) will in some way or another need an argument similar to the one used to prove (4.2). Recall the notation used for that purpose. We may in an analogous way as in (4.3) divide into cases whether $D_{T,n}$ and $D_{Y,n}$ occurs or not, to show that

$$T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}}) + \mu(\rho_{\nu(n)} - n) \xrightarrow{d} Y_{2M,i} - Y_+ + \mu r_+, \quad \text{as } n \rightarrow \infty,$$

Uniform integrability of $\left\{ \left(T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}}) + \mu(\rho_{\nu(n)} - n) \right)^2 \right\}_{n \geq 1}$, can be proved similar to the uniform integrability needed for (4.2). It follows that for $r = 1, 2$

$$\mathbb{E} \left[\left(T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}}) + \mu(\rho_{\nu(n)} - n) \right)^r \right] \rightarrow \mathbb{E} \left[(Y_{2M,i} - Y_+ + \mu r_+)^r \right],$$

as $n \rightarrow \infty$. From this we conclude that for the second term in the right-hand side of (4.5) we have that, for $C_i = \text{Var}(Y_{2M,i} - Y_+ + \mu r_+)$,

$$\text{Var} \left(T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}}) + \mu(\rho_{\nu(n)} - n) \right) = C_i + o(1), \quad \text{as } n \rightarrow \infty.$$

To tackle $\text{Cov} \left(T(\hat{v}_{\rho_{\nu(n)}}), \rho_{\nu(n)} \right)$, introduce $r_n := M + \max\{m_{n,k} < n : A_{m_{n,k}} \text{ occurs}\}$, and rewrite

$$\begin{aligned} \text{Cov} \left(T(\hat{v}_{\rho_{\nu(n)}}), \rho_{\nu(n)} \right) &= \text{Cov} \left(T(\hat{v}_{\rho_{\nu(n)}}) - T(\hat{v}_{r_n}), \rho_{\nu(n)} - n \right) \\ &\quad + \text{Cov} \left(T(\hat{v}_{r_n}), \rho_{\nu(n)} - n \right). \end{aligned}$$

It is easy to see that $\rho_{\nu(n)} - n \xrightarrow{d} r_+$ for $n \geq \rho_I$. Partitioning on whether $D_{T,n}$ and $D_{Y,n}$ occur or not, we see that, as $n \rightarrow \infty$,

$$T(\hat{v}_{\rho_{\nu(n)}}) - T(\hat{v}_{r_n}) \xrightarrow{d} Y_+, \quad \text{and} \quad \left(T(\hat{v}_{\rho_{\nu(n)}}) - T(\hat{v}_{r_n}) \right) (\rho_{\nu(n)} - n) \xrightarrow{d} Y_+ r_+.$$

Uniform integrability of $\{(\rho_{\nu(n)} - n)^2\}_{n \geq 1}$ and $\{(T(\hat{v}_{\rho_{\nu(n)}}) - T(\hat{v}_{r_n}))^2\}_{n \geq 1}$ is possible to deduce in a similar way as before, conditioning on $\Lambda_k = \{\rho_{\nu(n)} - r_n = k\}$. This implies that also $\{(T(\hat{v}_{\rho_{\nu(n)}}) - T(\hat{v}_{r_n}))(\rho_{\nu(n)} - n)\}_{n \geq 1}$ is uniformly integrable. We conclude that, as $n \rightarrow \infty$

$$\text{Cov} \left(T(\hat{v}_{\rho_{\nu(n)}}) - T(\hat{v}_{r_n}), \rho_{\nu(n)} - n \right) \rightarrow \mathbb{E}[Y_+ r_+] - \mathbb{E}[Y_+] \mathbb{E}[r_+] = \text{Cov}(Y_+, r_+),$$

On the event $D_{T,n}$, $T(\hat{v}_{r_n})$ depends on passage times below level n , but not on Δ , whereas $\rho_{\nu(n)} - n$ is independent of passage times below level n , and hence on $D_{T,n}$. It follows that

$$\mathbb{E} \left[T(\hat{v}_{r_n})(\rho_{\nu(n)} - n) 1_{D_{T,n}} \right] = \mathbb{E} \left[T(\hat{v}_{r_n}) 1_{D_{T,n}} \right] \mathbb{E}[r_+].$$

In particular,

$$\begin{aligned} \text{Cov} \left(T(\hat{v}_{r_n}), \rho_{\nu(n)} - n \right) &= \mathbb{E} \left[T(\hat{v}_{r_n})(\rho_{\nu(n)} - n) 1_{D_{T,n}} \right] - \mathbb{E} \left[T(\hat{v}_{r_n}) 1_{D_{T,n}} \right] \mathbb{E}[r_+] \\ &\quad + \mathbb{E} \left[T(\hat{v}_{r_n})(\rho_{\nu(n)} - n) 1_{D_{T,n}^c} \right] - \mathbb{E} \left[T(\hat{v}_{r_n}) 1_{D_{T,n}^c} \right] \mathbb{E}[r_+] \\ &= \mathbb{E} \left[T(\hat{v}_{r_n})(\rho_{\nu(n)} - n - \mathbb{E}[r_+]) 1_{D_{T,n}^c} \right]. \end{aligned}$$

As $n \rightarrow \infty$, this expression vanishes, since we can find an upper bound on $E[T(\hat{v}_{r_n})(\rho_{\nu(n)} - n - E[r_+])]$ in a similar way as before. We conclude that

$$\text{Cov}\left(T(\hat{v}_{\rho_{\nu(n)}}), \rho_{\nu(n)}\right) = \text{Cov}(Y_+, r_+) + o(1), \quad \text{as } n \rightarrow \infty.$$

The term $\text{Cov}(T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}, T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}}))$ is the only one left in the right-hand side of (4.5) to take care of. An application of the first part of Wald's lemma, we get

$$E[T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}] = E[T(\hat{v}_{\rho_0}) - \mu\rho_0].$$

The sequence $\{\tau_{S_k} - \mu S_k\}_{k=1}^{\nu(n)}$ has until now been considered as a sequence started with at $k = 1$ and stopped at $k = \nu(n)$. But, we can as well see it as a sequence in the opposite direction. That is, as a sequence started at the first point of regeneration after level $n + M$, and that is stopped at the first point of regeneration after level ρ_I . Let $T^* := T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}})$. On the event $\{\nu(n) \geq 1\}$, $T^* = T(\hat{v}_{\rho_{\nu(n)-1}}, v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)-1}}, \hat{v}_{\rho_{\nu(n)}})$ and is independent of $\tau_{S_k} - \mu S_k$ for $k < \nu(n)$. The event $\{\nu(n) \geq 1\}$ is itself independent of $\{\tau_{S_k} - \mu S_k\}_{k \geq 1}$. This allows us to apply the third part of Wald's lemma to obtain

$$\begin{aligned} E[(T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)})T^*1_{\{\nu(n) \geq 1\}}] &= E[(\tau_{S_{\nu(n)}} - \mu S_{\nu(n)})T^*1_{\{\nu(n) \geq 1\}}] \\ &\quad + E[(T(\hat{v}_{\rho_0}) - \mu\rho_0)T^*1_{\{\nu(n) \geq 1\}}]. \end{aligned}$$

Since $E[(T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)})T^*1_{\{\nu(n)=0\}}] = E[(T(\hat{v}_{\rho_0}) - \mu\rho_0)T^*1_{\{\nu(n)=0\}}]$, we obtain

$$E[(T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)})T^*] = E[(\tau_{S_{\nu(n)}} - \mu S_{\nu(n)})T^*1_{\{\nu(n) \geq 1\}}] + E[(T(\hat{v}_{\rho_0}) - \mu\rho_0)T^*],$$

and, in particular,

$$\begin{aligned} \text{Cov}(T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}, T^*) &= E[(\tau_{S_{\nu(n)}} - \mu S_{\nu(n)})T^*1_{\{\nu(n) \geq 1\}}] \\ &\quad + \text{Cov}(T(\hat{v}_{\rho_0}) - \mu\rho_0, T^*). \end{aligned} \tag{4.6}$$

Let $r_- := M + \max\{n_k < 0 : A_{n_k} \text{ occurs}\}$, $Y_- := T(\hat{v}_{r_-}, \hat{v}_{r_+})$ and $Z_{k,i} := T(\hat{v}_{r_-}, v_{k,i})$. Observe that

$$(\tau_{S_{\nu(n)}} - \mu S_{\nu(n)})T^*1_{\{\nu(n) \geq 1\}} \stackrel{d}{=} (Y_- - \mu(r_+ - r_-))(Z_{2M,i} - Y_-)1_H,$$

where $H = \{A_{n_k} \text{ occurs for some } \rho_I + \Delta - n \leq n_k < 0\}$. To conclude that the former term in the right-hand side of (4.6) converges as $n \rightarrow \infty$ can now be done via the Monotone convergence theorem. That the limit is finite can be seen in a similar way as before, conditioning on $\Lambda_k = \{r_+ - r_- = k\}$. (Note that $Y_- \stackrel{d}{=} \tau_{S_k}$, and $r_+ - r_- \stackrel{d}{=} S_k$.) For the latter term in the right-hand side of (4.6), let $Z'_{2M,i}$ and Y'_+ be defined in the same way as $Z_{2M,i}$ and Y_+ above, but now for a set of passage times $\{\tau'_e\}_{e \in \mathbb{E}}$ independent of $\{\tau_e\}_{e \in \mathbb{E}}$ (that defines $T(\hat{v}_{\rho_0}) - \mu\rho_0$), but with the same Δ . By conditioning on the events $\{\nu(n) \geq 1\}$ (with respect to $\{\tau_e\}_{e \in \mathbb{E}}$) and H (with respect to $\{\tau'_e\}_{e \in \mathbb{E}}$), we see that as $n \rightarrow \infty$

$$(T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}})) \xrightarrow{d} (Z'_{2M,i} - Y'_+),$$

and

$$(T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}}))(T(\hat{v}_{\rho_0}) - \mu\rho_0) \xrightarrow{d} (Z'_{2M,i} - Y'_+)(T(\hat{v}_{\rho_0}) - \mu\rho_0).$$

Since $\{(T(\hat{v}_{\rho_0}) - \mu\rho_0)^2\}_{n \geq 1}$ and $\{(T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}}))^2\}_{n \geq 1}$ can be seen to be uniformly integrable, as above, $\{(T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}}))(T(\hat{v}_{\rho_0}) - \mu\rho_0)\}_{n \geq 1}$ is also uniform integrable, and we have that

$$\text{Cov}(T(\hat{v}_{\rho_0}) - \mu\rho_0, T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}})) \rightarrow \text{Cov}(T(\hat{v}_{\rho_0}) - \mu\rho_0, Y'_{2M} - Y'_+), \quad \text{as } n \rightarrow \infty.$$

That is, for some constant C_i ,

$$\text{Cov}(T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}, T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}})) = C_i + o(1), \quad \text{as } n \rightarrow \infty.$$

This all together amount to that there exists a finite constant C_i such that

$$\text{Var}(T(v_{n+2M,i})) = \sigma^2 n + C_i + o(1), \quad \text{as } n \rightarrow \infty,$$

which proves the theorem. \square

5 Geodesics and time constants

The path along which an infection travels from one vertex to another for first-passage percolation on essentially 1-dimensional periodic graphs is studied in this section. The existence of such minimising paths can be easily derived from Lemma 2.3.

Proposition 5.1. *Let U and V be two finite sets of vertices of an essentially 1-dimensional periodic graph. There is an almost surely finite path γ from U to V , such that*

$$T(\gamma) = T(U, V).$$

Moreover, if the passage-time distribution does not have any point masses, then γ is almost surely unique.

It follows directly from the statement that for any finite I , n and v , there are almost surely finite paths attaining the infima in T_n , $T(v)$ and $\max_{v \in \mathbb{V}_{\mathcal{G}_n}} T(v)$.

Proof. We may assume that $U \cup V \subseteq \bigcup_{k=0}^m \mathbb{V}_{\mathcal{G}_k}$. Assume further that t' , t'' and M are chosen in accordance with Lemma 2.3. With probability one the event $A_{m+n} \cap A_{-2M-n}$ will occur for infinitely many $n \geq 0$. Let l be the least such n . It follows from Lemma 2.3 that for any path Γ between u and v that reach beyond level $m+l+2M$ in the positive direction, or level $-2M-l$ in the negative direction, there is another path Γ' that only visits vertices in $\bigcup_{k=-2M-l}^{m+l+2M} \mathbb{V}_{\mathcal{G}_k}$, and that satisfies $T(\Gamma) \geq T(\Gamma')$. Thus, since there are only finitely many edges between level $-2M-l$ and $m+l+2M$, it follows that $T(U, V)$ is the minimum of the passage times over an almost surely finite number of paths. This proves the first statement. The second statement also follows from this, together with the fact that the probability of two paths having the same passage time is zero, when the passage-time distribution is free of point masses. \square

As in the introduction, we will use the term *geodesic* to refer to a path attaining the minimal passage time between two vertices, or two finite sets of vertices. Since geodesics are not necessarily unique, we assume a fixed deterministic rule to choose one when several are possible, e.g. the shortest (with some additional rule for breaking ties). Let $\gamma(u, v)$ denote the geodesic between u and v . Several properties of geodesics can be investigated. We will in what comes mainly consider the length of geodesics.

Let $N(u, v)$ denote the length of $\gamma(u, v)$. The regenerative behaviour studied in Section 2 will again play an important rôle. It follows from Lemma 2.3 that any geodesic from $u \in \mathbb{V}_{\mathcal{G}_n}$ to $v \in \mathbb{V}_{\mathcal{G}_m}$ (where $n \leq m$) passes \hat{v}_{ρ_k} , for all $n + M \leq \rho_k \leq m - M$. Moreover, $\{N(\hat{v}_{\rho_{k-1}}, \hat{v}_{\rho_k})\}_{k \in \mathbb{Z}}$ forms an i.i.d. sequence, which we may use to write

$$N(\hat{v}_{\rho_n}) = N(\hat{v}_{\rho_0}) + \sum_{k=1}^n N(\hat{v}_{\rho_{k-1}}, \hat{v}_{\rho_k}).$$

It is now easy to see that $\{N(\hat{v}_n)\}_{n \geq 1}$ is a regenerative sequence, with sequence of regenerative levels $\{\rho_n\}_{n \geq 0}$. Since there are only finitely many vertices at each level, say K , it follows that

$$N(\hat{v}_{\rho_{k-1}}, \hat{v}_{\rho_k}) \leq KS_k,$$

for each $k \in \mathbb{Z}$. In particular, $N(\hat{v}_{\rho_{k-1}}, \hat{v}_{\rho_k})$ has finite moments of all orders. Set

$$\alpha := \frac{\mathbb{E}[N(\hat{v}_{\rho_{k-1}}, \hat{v}_{\rho_k})]}{\mu_S}, \quad \text{and} \quad \sigma_N^2 := \frac{\text{Var}(N(\hat{v}_{\rho_{k-1}}, \hat{v}_{\rho_k}) - \alpha S_k)}{\mu_S}. \quad (5.1)$$

Trivially, $\alpha \geq 1$ for any essentially 1-dimensional periodic graph.

Recall that we let $\{\hat{N}_n\}_{n \geq 1}$ denote either of the sequences $\{N_n\}_{n \geq 1}$, $\{\max_{v \in \mathbb{V}_{\mathcal{G}_n}} N(v)\}_{n \geq 1}$ and $\{N(v_n)\}_{n \geq 1}$, where $\{v_n\}_{n \geq 1}$ is any sequence of vertices such that v_n is at level n . By mimicking the proofs of Theorem 1.3, 1.4, 1.5, 1.6, 1.7 and 3.6, one may prove the following two results. Note that no moment conditions are required, since $N(\hat{v}_{\rho_{k-1}}, \hat{v}_{\rho_k})$ has finite moments of all orders. The adaptations of the proofs are left to the reader.

Theorem 5.2. *Consider first-passage percolation on any essentially 1-dimensional periodic graph \mathcal{G} , with any passage-time distribution that do not concentrate all mass to a single point. Then, the statements of Theorem 1.3, 1.4, 1.5 and 3.6 holds (with constants α and σ_N^2) for the sequence $\{\hat{N}_n\}_{n \geq 1}$.*

Theorem 1.8 is included in Theorem 5.2.

Theorem 5.3. *Consider first-passage percolation on any essentially 1-dimensional periodic graph \mathcal{G} , with any passage-time distribution that do not concentrate all mass to a single point. Let $v_{n,i}$ be a specific vertex at level n . Then, for some $C_i, C'_i \in \mathbb{R}$, as $n \rightarrow \infty$,*

$$\begin{aligned} \mathbb{E}[N(v_{n,i})] &= \alpha n + C_i + o(1), \\ \text{Var}(N(v_{n,i})) &= \sigma_N^2 n + C'_i + o(1). \end{aligned}$$

Geodesics are, as seen via Lemma 2.3, locally determined. Thus, it makes sense to talk about an infinite geodesic from $-\infty$ to ∞ . Let to this end γ^* denote the unique (subject to the rule for breaking ties) path that between level ρ_{k-1} and ρ_k coincides with $\gamma(\hat{v}_{\rho_{k-1}}, \hat{v}_{\rho_k})$, for each $k \in \mathbb{Z}$. The resulting infinite path is indeed a geodesic, i.e., any finite portion $\tilde{\gamma}^*$ of γ^* with endpoints u and v satisfies $T(\tilde{\gamma}^*) = T(u, v)$. It is possible to characterize time and length constants in terms of the infinite geodesic.

Proposition 5.4.

$$\begin{aligned}\alpha &= \sum_{v \in \mathbb{V}_{\mathcal{G}_0}} P(v \in \gamma^*) = \sum_{e \in \mathbb{E}_{\mathcal{G}_0}^*} P(e \in \gamma^*), \\ \mu &= \sum_{e \in \mathbb{E}_{\mathcal{G}_0}^*} \mathbb{E}[\tau_e 1_{\{e \in \gamma^*\}}] = \sum_{e \in \mathbb{E}_{\mathcal{G}_0}^*} \mathbb{E}[\tau_e | e \in \gamma^*] P(e \in \gamma^*).\end{aligned}$$

Proof. We will deduce the characterization of α in terms of vertices, and leave the remaining cases, which are deduced similarly, to the reader. Observe that

$$N(u, w) = \sum_{k \in \mathbb{Z}} \sum_{v \in \mathbb{V}_{\mathcal{G}_k}} 1_{\{v \in \gamma(u, w)\}} - 1.$$

According to Theorem 5.2, we have

$$\frac{\mathbb{E}[N(\hat{v}_{-n}, \hat{v}_n)]}{2n} \rightarrow \alpha, \quad \text{as } n \rightarrow \infty.$$

Define $N^* := \sum_{k=-n+\sqrt{n}}^{n-\sqrt{n}} \sum_{v \in \mathbb{V}_{\mathcal{G}_k}} 1_{\{v \in \gamma^*\}}$. Clearly

$$\frac{\mathbb{E}[N^*]}{2n} = \frac{2(n - \sqrt{n})}{2n} \sum_{v \in \mathbb{V}_{\mathcal{G}_0}} P(v \in \gamma^*) \rightarrow \sum_{v \in \mathbb{V}_{\mathcal{G}_0}} P(v \in \gamma^*), \quad \text{as } n \rightarrow \infty,$$

so we are finished if we show that $\mathbb{E}[|N(\hat{v}_{-n}, \hat{v}_n) - N^*|]/n \rightarrow 0$, as $n \rightarrow \infty$. Let

$$D_n := \{A_k \cap A_{-k-2M} \text{ occurs for some } k \in [n - \sqrt{n}, n - 2M]\},$$

where A_k and M are as defined in Section 2. Let $\kappa_n := \min\{k \geq n : A_k \cap A_{-k-2M} \text{ occurs}\}$. Trivially, $|N(\hat{v}_{-n}, \hat{v}_n) - N^*| \leq 4|\mathbb{V}_{\mathcal{G}_0}|(\kappa_n + 2M)$. On the event D_n we have

$$\begin{aligned}N(\hat{v}_{-n}, \hat{v}_n) - N^* &= \sum_{\substack{k > n - \sqrt{n} \\ k < -n + \sqrt{n}}} \sum_{v \in \mathbb{V}_{\mathcal{G}_k}} 1_{\{v \in \gamma(u, w)\}} - 1 \\ &\leq 2|\mathbb{V}_{\mathcal{G}_0}|(\kappa_n + 2M - n + \sqrt{n}).\end{aligned}$$

Since $\kappa_n - n$ can be dominated by a geometrically distributed random variable, similar to S_k in the proof of the first part of Proposition 2.7, we easily realise that

$$\begin{aligned}\mathbb{E}[|N(\hat{v}_{-n}, \hat{v}_n) - N^*|] &= \mathbb{E}[|N(\hat{v}_{-n}, \hat{v}_n) - N^*|(1_{D_n} + 1_{D_n^c})] \\ &\leq 2|\mathbb{V}_{\mathcal{G}_0}|(\mathbb{E}[\kappa_n - n] + 2M + \sqrt{n}) + 4|\mathbb{V}_{\mathcal{G}_0}|(\mathbb{E}[\kappa_n] + 2M)P(D_n^c) \\ &\leq 4|\mathbb{V}_{\mathcal{G}_0}|(C + \sqrt{n} + nP(D_n^c)) = o(n).\end{aligned}$$

As mentioned, the remaining characterizations are deduced similarly. \square

Benjamini et al. (2003) posed the question whether for first-passage percolation on the \mathbb{Z}^d lattice, $P(\mathbf{0} \in \gamma(-\mathbf{n}, \mathbf{n})) \rightarrow 0$ as $n \rightarrow \infty$ (given existence of geodesics). One may pose a corresponding question for first-passage percolation on the (K, d) -tube. Let γ_K^* denote the infinite geodesic on the (K, d) -tube. How does $P(v \in \gamma_K^*)$ behave as $K \rightarrow \infty$? In particular, does

$$\max_{v \in \mathbb{V}_{\mathcal{G}_n}} P(v \in \gamma_K^*) \rightarrow 0, \quad \text{as } K \rightarrow \infty?$$

If it does, at which rate? Let α_K denote the constant α associated to the (K, d) -tube. By symmetry it is easily realised that for even K ,

$$\max_{v \in \mathbb{V}_{\mathcal{G}_n}} P(v \in \gamma_K^*) \leq \frac{\alpha_K}{2^{d-1}}.$$

We do not have enough symmetry to conclude a similar upper bound in K . We can increase the symmetry of the $(K, 2)$ -tube by connecting the vertices $(n, 0)$ and $(n, K-1)$, for each n , by an edge. On the resulting graph we have, for every vertex v ,

$$P(v \in \gamma^*) = \frac{\alpha}{K}.$$

The same can be done for any (K, d) -tube. Join, for each $j = 2, 3, \dots, d$, the vertices

$$(m_1, m_2, \dots, m_{j-1}, 0, m_{j+1}, \dots, m_{K-1})$$

and

$$(m_1, m_2, \dots, m_{j-1}, K-1, m_{j+1}, \dots, m_{K-1})$$

by an edge, for all $m_1 \in \mathbb{Z}$ and $m_2, \dots, m_{K-1} \in \{0, 1, \dots, K-1\}$. Refer to the resulting graph as a (K, d) -cylinder. Let $\tilde{\alpha}_K$ denote the constant α associated to the (K, d) -cylinder. We have for every vertex v of the (K, d) -cylinder

$$P(v \in \gamma^*) = \frac{\tilde{\alpha}_K}{K^{d-1}}.$$

Thus, in view of Remark 5.5 below, there is a constant $C = C(d)$ such that

$$\frac{1}{K^{d-1}} \leq P(v \in \gamma^*) \leq \frac{C}{K^{d-1}},$$

for every $K \geq 1$, and every vertex v of the (K, d) -cylinder.

Remark 5.5. Provided that $E[\tau_e] < \infty$ and that $P_\tau(0)$ is sufficiently small, Kesten (1986) gives an argument that shows that on the \mathbb{Z}^d lattice, there is a constant $C = C(d)$ such that $E[N(u, v)] \leq C \text{dist}(u, v)$ for all vertices u and v (cf. Howard (2004, page 146)). It is clear that the argument also applies to (K, d) -tubes and (K, d) -cylinders. That is, on either of these graphs, there exists a $C = C(d)$ such that for all $K \geq 1$

$$E[N(u, v)] \leq C \text{dist}(u, v)$$

for all vertices u and v . A direct consequence of this is that $\alpha_K \leq C$ and $\tilde{\alpha}_K \leq C$, for some finite constant C , for all $K \geq 1$. \square

Remark 5.6. An object closely related to geodesics is the *tree of infection* Ψ . Let $v_0 \in \mathbb{V}_{\mathcal{G}_0}$ denote a vertex referred to as the origin. The tree of infection is then defined as the tree $\Psi = \bigcup_{v \in \mathbb{V}} \gamma(v_0, v)$ spanning the underlying graph \mathcal{G} (see Figure 2, page 11 for a realisation on the $(2, 2)$ -tube). One may ask for the number of infinite self-avoiding paths in Ψ started at the origin, denoted by $\kappa(\Psi)$. On any essentially 1-dimensional periodic graph $\kappa(\Psi) = 2$, almost surely. To see this, for any $M \geq 1$, let κ_M denote the number of self-avoiding paths in Ψ that reach level M . With probability one, for some $n \geq M$ the event A_n will occur. It follows from Lemma 2.3 that the geodesic from u to v , for all u at level n and v at level $n + 2M$, have all to pass a certain vertex at level $n + M$. Thus, only one of the κ_M self-avoiding paths in Ψ will survive beyond level $n + 2M$. This implies that precisely one self-avoiding path will reach infinitely far in positive direction. The same applies in negative direction. From this we conclude that $\kappa(\Psi) = 2$ almost surely.

On the \mathbb{Z}^d lattice for $d \geq 2$, it is believed that $\kappa(\Psi)$ is infinite. So far, it is only known that $\kappa(\Psi) \geq 2d$ almost surely (see Hoffman (2008) and Gour  r   (2007)). It would be interesting to prove that $\kappa(\Psi)$ is almost surely constant. That would follow from an higher dimensional version of the Proposition 1.10. It is not known whether such a coupling is possible. \square

5.1 Continuity of constants

The following result is inspired by a similar result due to Cox (1980) and Cox and Kesten (1981), who in their case consider first-passage percolation on the \mathbb{Z}^d lattice. The proof of the lattice case is rather lengthy. Due to the regenerative behaviour in the case of essentially 1-dimensional periodic graphs, and in particular the characterization of μ and σ given in (2.7), and of α and σ_N given in (5.1), the proof of our result turns out to be much simpler.

Proposition 5.7. *Let F_m for $m = 1, 2, \dots, \infty$ be distribution functions such that $F_m \xrightarrow{d} F_\infty$ as $m \rightarrow \infty$. Then, as $m \rightarrow \infty$,*

$$\alpha(F_m) \rightarrow \alpha(F_\infty) \quad \text{and} \quad \sigma_N(F_m) \rightarrow \sigma_N(F_\infty).$$

Assume further that there are $p \geq 1$ (edge) disjoint paths from \hat{v}_0 to \hat{v}_1 , and a distribution function V such that $F_m \geq V$ for all $m \geq 1$. Let Y_V denote the minimum of p independent random variables with distribution V . If, in addition, $E[Y_V] < \infty$ and $E[Y_V^2] < \infty$, then as $m \rightarrow \infty$, respectively,

$$\mu(F_m) \rightarrow \mu(F_\infty) \quad \text{and} \quad \sigma(F_m) \rightarrow \sigma(F_\infty).$$

Remark 5.8. This result will be used in Ahlberg (2011) in order to prove a dynamically stable version of Theorem 1.3. \square

In order to compare the different distributions, we will use a coupling of random variables via their inverse distribution functions $F^{-1}(u) := \inf\{x \in \mathbb{R} : F(x) \geq u\}$. The same approach is used in Cox (1980) and Cox and Kesten (1981). Indeed, if U is uniformly distributed on $[0, 1]$, then $F^{-1}(U)$ has distribution F , since

$$P(F^{-1}(U) \leq x) = P(U \leq F(x)) = F(x), \quad \text{for all } x \in \mathbb{R}.$$

Thus, if we let F run over the class of distribution functions, then $\{F^{-1}(U)\}_F$ generates a coupling of all differently distributed random variables. Note that F^{-1} is nondecreasing since

F is, and has at most countably many discontinuity points (since \mathbb{Q} is countable, and for each discontinuity point u , $[F^{-1}(u-), F^{-1}(u)] \cap \mathbb{Q} \neq \emptyset$). It is not hard to prove that (see e.g. (Thorisson, 2000, Section 1.8.4)), as $m \rightarrow \infty$, $F_m \xrightarrow{d} F_\infty$ implies $F_m^{-1}(u) \rightarrow F_\infty^{-1}(u)$ for all continuity points $u \in (0, 1)$. In particular, $F_m^{-1}(U) \rightarrow F_\infty^{-1}(U)$ almost surely, as $m \rightarrow \infty$.

Once we have the above coupling, the rest will follow fairly easily. For i.i.d. sequences, if $F_m \xrightarrow{d} F_\infty$, $F_m \geq V$ for all m , and V has finite mean, then $F_m^{-1}(U) \leq V^{-1}(U)$ and $\mathbb{E}[F_m^{-1}(U)] \rightarrow \mathbb{E}[F_\infty^{-1}(U)]$ as $m \rightarrow \infty$, by the Dominated convergence theorem. For the proof of the proposition, the idea is similar.

Proof of Proposition 5.7. Let $\{U_e\}_{e \in \mathbb{E}}$ be a collection of independent random variables uniformly distributed on $[0, 1]$. Thus, as F ranges over the class of passage-time distributions, then $\{\{F^{-1}(U_e)\}_{e \in \mathbb{E}}\}_F$ simultaneously couples i.i.d. sets of passage times of the graph. Choose $a \in (0, 1/2)$ such that $F_\infty^{-1}(1-a) > F_\infty^{-1}(a)$, and F_∞^{-1} is continuous in both a and $1-a$. Take $\epsilon > 0$ such that $F_\infty^{-1}(1-a) - F_\infty^{-1}(a) > 2\epsilon$. Choose $L < \infty$ such that

$$|F_m^{-1}(a) - F_\infty^{-1}(a)| \leq \epsilon \quad \text{and} \quad |F_m^{-1}(1-a) - F_\infty^{-1}(1-a)| \leq \epsilon,$$

for all $m \geq L$. Recall the definition of $A_n = A_n(M, t', t'')$ in Section 2. Set $t' = F_\infty^{-1}(a) + \epsilon$ and $t'' = F_\infty^{-1}(1-a) - \epsilon$, and let M be chosen in accordance with Lemma 2.3 (recall that M is chosen independently of the passage time distribution F_m). For the same M (and with notation as in Section 2), define

$$\tilde{A}_n = \tilde{A}_n(M) := \left\{U_e \leq a, \forall e \in \hat{E}_n\right\} \cap \left\{U_e \geq 1-a, \forall e \in E_n \setminus \hat{E}_n\right\}.$$

Since $a > 0$, we have $P(\tilde{A}_n) > 0$. For all $m \geq L$ we have

$$\begin{cases} F_m^{-1}(u) \leq t', & \text{for } u \leq a, \\ F_m^{-1}(u) \geq t'', & \text{for } u \geq 1-a. \end{cases}$$

With a slight abuse of notation, we let $A_n(F)$ denote the event A_n with respect to $\{F^{-1}(U_e)\}_{e \in \mathbb{E}}$. In particular, this implies that $\tilde{A}_n \subseteq A_n(F_m)$ for all $L \leq m \leq \infty$. Define a sequence $\{\tilde{\rho}_k\}_{k \geq 0}$ with respect to \tilde{A}_n analogously as in Section 2. Note that for $m \geq L$, the sequence $\{\tilde{\rho}_k\}_{k \geq 0}$ is a subsequence of $\{\rho_k\}_{k \geq 0}$ defined with respect to $A_n(F_m)$. The advantage of this is that we get a regenerative sequence valid for all distributions F_m with $L \leq m \leq \infty$.

From here the result follows quickly. Let $T_F(u, v)$ and $N_F(u, v)$ denote the passage time and length of geodesic, respectively, between u and v with respect to $\{F^{-1}(U_e)\}_{e \in \mathbb{E}}$. For $m \geq L$ we have the characterization

$$\begin{aligned} \alpha(F_m) &= \frac{\mathbb{E}[N_{F_m}(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1})]}{\mathbb{E}[\tilde{\rho}_1 - \tilde{\rho}_0]}, & \sigma_N(F_m) &= \frac{\mathbb{E}[N_{F_m}^2(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1})]}{\mathbb{E}[\tilde{\rho}_1 - \tilde{\rho}_0]}, \\ \mu(F_m) &= \frac{\mathbb{E}[T_{F_m}(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1})]}{\mathbb{E}[\tilde{\rho}_1 - \tilde{\rho}_0]}, & \sigma(F_m) &= \frac{\mathbb{E}[T_{F_m}^2(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1})]}{\mathbb{E}[\tilde{\rho}_1 - \tilde{\rho}_0]}. \end{aligned}$$

Thus, in order to prove that $\mu(F_m) \rightarrow \mu(F_\infty)$ as $m \rightarrow \infty$, it suffices to show that

$$\mathbb{E}[T_{F_m}(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1})] \rightarrow \mathbb{E}[T_{F_\infty}(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1})], \quad \text{as } m \rightarrow \infty. \quad (5.2)$$

But $F_m^{-1}(U) \rightarrow F_\infty^{-1}(U)$ almost surely, as $m \rightarrow \infty$, and therefore also

$$T_{F_m}(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1}) \rightarrow T_{F_\infty}(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1}), \quad \text{almost surely.}$$

Since $T_V(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1})$ has finite mean when Y_V does (according to Proposition 2.7), and since $T_{F_m}(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1}) \leq T_V(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1})$, we conclude by the Dominated convergence theorem that (5.2) holds when $E[Y_V] < \infty$. To see that the domination is not necessary in order to prove convergence of $\alpha(F_m)$ to $\alpha(F_\infty)$, it suffices to realize that for $m \geq L$ and some $C < \infty$, we have $N_{F_m}(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1}) \leq C(\tilde{\rho}_1 - \tilde{\rho}_0)$. The remaining conclusions are drawn similarly. \square

Remark 5.9. The true condition for the convergence $E[F_m^{-1}(U)] \rightarrow E[F_\infty^{-1}(U)]$ is in fact uniform integrability of $\{F_m\}_{m \geq 1}$. In the same way it is possible to relax the moment condition on Y_V to uniform integrability of $\{Y_{F_m}^r\}_{m \geq 1}$ (for $r = 1$ or 2), which grants uniform integrability of $\{T_{F_m}^r(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1})\}_{m \geq 1}$. We leave it to the reader to go through the details. \square

5.2 Time constant and the (K, d) -tube

Let μ_K denote the time constant associated with the (K, d) -tube. It is easy to realize that $\mu_{K+1} \leq \mu_K$. However, it is hard to prove that without a (trivial) coupling argument. In fact, strict inequality holds, for which we will need the same coupling in order to see. The coupling is as follows. Let $\{\tau_e\}_{e \in \mathbb{E}_{\mathbb{Z}^d}}$ be i.i.d. passage times associated to the \mathbb{Z}^d lattice. The (K, d) -tubes are naturally seen as subgraphs of the \mathbb{Z}^d lattice. Let $T_K(u, v)$ denote the passage time with respect to $\{\tau_e\}_{e \in \mathbb{E}_{\mathbb{Z}^d}}$, between u and v , when only paths in the $\mathbb{Z} \times \{0, \dots, K-1\}^{d-1}$ nearest neighbour graph (the (K, d) -tube) are allowed. This produces a simultaneous coupling of the passage time on (K, d) -tubes for all $K \geq 1$. Trivially, $T_{K+1}(u, v) \leq T_K(u, v)$ for any vertices u and v in $\mathbb{Z} \times \{0, \dots, K-1\}^{d-1}$.

Proposition 5.10. $\mu_{K+1} < \mu_K$, for all $K \geq 1$.

Proof. Let A_n^K be the event defined in (2.4) with respect to the (K, d) -tube, for γ_n chosen to be the straight line segment between the points $(n, K, 0, \dots, 0)$ and $(n+2M, K, 0, \dots, 0)$. It follows from Lemma 2.3 that if A_n^{K+1} occurs, then

$$\delta := T_K(n\mathbf{e}_1, (n+2M)\mathbf{e}_1) - T_{K+1}(n\mathbf{e}_1, (n+2M)\mathbf{e}_1) > 0.$$

Thus, if $m_k = (2M+1)k$, then

$$T_{K+1}(0\mathbf{e}_1, m_k\mathbf{e}_1) + \delta \sum_{j=0}^{k-1} 1_{A_{m_j}^{K+1}} \leq T_K(0\mathbf{e}_1, m_k\mathbf{e}_1)$$

for all $k \geq 0$. Dividing by m_k and taking limits as k tends to infinity, gives

$$\mu_{K+1} + \delta P(A_n^{K+1})(2M+1)^{-1} \leq \mu_K. \quad \square$$

In order to prove that the limit of the sequence $\{\mu_K\}_{K \geq 1}$ is $\mu(\mathbf{e}_1)$, i.e., equals the time constant for the \mathbb{Z}^d lattice, we will use a coupling similar to the above one. For $K = 0, 1, \dots, \infty$,

let $\tilde{T}_K(u, v)$ denote the passage time with respect to $\{\tau_e\}_{e \in \mathbb{E}_{\mathbb{Z}^d}}$, between u and v , when only paths in the $\mathbb{Z} \times \{-K, \dots, K\}^{d-1}$ nearest neighbour graph (the $(2K+1, d)$ -tube) are allowed. This produces a simultaneous coupling of the passage time on (K, d) -tubes for odd K . The case $K = \infty$ corresponds to the \mathbb{Z}^d lattice.

Proposition 5.11. $\lim_{K \rightarrow \infty} \mu_K = \mu(\mathbf{e}_1)$.

Proof. Clearly $\tilde{T}_K(0\mathbf{e}_1, n\mathbf{e}_1) \geq \tilde{T}_{K+1}(0\mathbf{e}_1, n\mathbf{e}_1)$. For all n we get

$$\tilde{T}_\infty(0\mathbf{e}_1, n\mathbf{e}_1) = \lim_{K \rightarrow \infty} \tilde{T}_K(0\mathbf{e}_1, n\mathbf{e}_1) = \inf_{K \geq 0} \tilde{T}_K(0\mathbf{e}_1, n\mathbf{e}_1), \quad \text{almost surely.}$$

An application of the Monotone convergence theorem

$$\mathbb{E}[\tilde{T}_\infty(0\mathbf{e}_1, n\mathbf{e}_1)] = \lim_{K \rightarrow \infty} \mathbb{E}[\tilde{T}_K(0\mathbf{e}_1, n\mathbf{e}_1)] = \inf_{K \geq 0} \mathbb{E}[\tilde{T}_K(0\mathbf{e}_1, n\mathbf{e}_1)].$$

Since $\exists \lim_{n \rightarrow \infty} a_n/n = \inf_{n \geq 1} a_n/n$, for any subadditive real-valued sequence $\{a_n\}_{n \geq 1}$, we have for any $0 \leq K \leq \infty$ that

$$\mu_{2K+1} = \lim_{n \rightarrow \infty} \frac{\mathbb{E}[\tilde{T}_K(0\mathbf{e}_1, n\mathbf{e}_1)]}{n} = \inf_{n \geq 1} \frac{\mathbb{E}[\tilde{T}_K(0\mathbf{e}_1, n\mathbf{e}_1)]}{n}.$$

Thus, since μ_K is non-increasing in K

$$\begin{aligned} \lim_{K \rightarrow \infty} \mu_{2K+1} &= \inf_{K \geq 0} \inf_{n \geq 1} \frac{\mathbb{E}[\tilde{T}_K(0\mathbf{e}_1, n\mathbf{e}_1)]}{n} = \inf_{n \geq 1} \inf_{K \geq 0} \frac{\mathbb{E}[\tilde{T}_K(0\mathbf{e}_1, n\mathbf{e}_1)]}{n} \\ &= \inf_{n \geq 1} \frac{\mathbb{E}[\tilde{T}_\infty(0\mathbf{e}_1, n\mathbf{e}_1)]}{n} = \mu(\mathbf{e}_1). \end{aligned} \quad \square$$

6 Exact coupling and a 0–1 law

The aim for this section is to couple first-passage percolation infections with different initial configurations, i.e., different initially infected components, in such a way that the infections will eventually coincide. As an application of this, we shall prove a 0–1 law. The method of proof will once again make use of the regenerative behaviour explored in Section 2.

First we must state what we mean by a coupling. A *coupling* of two random variables $X \sim P$ and $Y \sim P'$ on a measurable space (E, \mathcal{E}) , is a joint distribution \hat{P} of (X, Y) , i.e., a measure on (E^2, \mathcal{E}^2) , such that its marginal distributions coincide with P and P' , respectively. When we couple two time-dependent random elements $\{X_t\}_{t \geq 0}$ and $\{Y_t\}_{t \geq 0}$, we say that the coupling is *exact* if with probability one there exists a $T_c < \infty$ such that $X_t = Y_t$, for all $t \geq T_c$.

We will present an exact coupling of the sets of infected vertices B_t and B'_t of two first-passage percolation processes with different initial configurations. Recall that we let $P_\tau(\cdot)$ denote the distribution of τ_e , and let \mathcal{R}_+ denote the Borel σ -algebra on $[0, \infty)$. Then $\{\tau_e\}_{e \in \mathbb{E}}$ and $\{\tau'_e\}_{e \in \mathbb{E}}$ are random elements on the product space $([0, \infty)^{\mathbb{E}}, \mathcal{R}_+^{\mathbb{E}})$, each with distribution given by the product measure $P_\tau^{\mathbb{E}}$. Let \mathbb{E}_n denote the set of edges between level $-n$ and n , but not including edges between two vertices at level $-n$ and n . In the same manner \mathbb{E}_n^c denotes the set of edges at and before level $-n$, as well as at level n and beyond.

We shall prove the following result which is slightly stronger than Proposition 1.10.

Proposition 6.1 (Coupling, continuous times). *Let I and I' be finite subsets of the set of vertices \mathbb{V} of an essentially 1-dimensional periodic graph \mathcal{G} . Assume that the passage time distribution P_τ has an absolutely continuous component (with respect to Lebesgue measure). For any $m \geq 0$, there exists a coupling of $\{\tau_e\}_{e \in \mathbb{E}_m^c}$ and $\{\tau'_e\}_{e \in \mathbb{E}_m^c}$ such that if $\{\tau_e\}_{e \in \mathbb{E}_m}$ and $\{\tau'_e\}_{e \in \mathbb{E}_m}$ each have distribution $P_\tau^{\mathbb{E}_m}$, then the marginal distributions of $\{\tau_e\}_{e \in \mathbb{E}}$ and $\{\tau'_e\}_{e \in \mathbb{E}}$ are given by the product measure $P_\tau^{\mathbb{E}}$, and such that if first-passage percolation is performed with $(I, \{\tau_e\}_{e \in \mathbb{E}})$ and $(I', \{\tau'_e\}_{e \in \mathbb{E}})$, respectively, then with probability one there exists an $N_c < \infty$ and a $T_c < \infty$, such that*

$$T(v_n) = T'(v_n) \quad \text{and} \quad B_t = B'_t, \quad (6.1)$$

for all $v_n \in \mathbb{V}_{\mathcal{G}_n}$ for $n \geq N_c$, and for all $t \geq T_c$.

When the passage time distribution P_τ is discrete, i.e., $P_\tau(\Lambda) = 1$ for the set of point masses

$$\Lambda := \{t_j \in [0, \infty) : P_\tau(t_j) > 0\},$$

the statement of Proposition 6.1 is not true in general. More precisely, there are essentially 1-dimensional periodic graphs on which no exact coupling is possible (cf. Remark 6.6). In the discrete case, we will therefore restrict our attention to the case of (K, d) -tubes.

Proposition 6.2 (Coupling, discrete times). *Let I and I' be finite subsets of the set of vertices \mathbb{V} of the (K, d) -tube, for $K, d \geq 2$. Assume that the passage time distribution P_τ is such that $P_\tau(\Lambda) = 1$ for the set of point masses Λ and that either of the following hold:*

a) *there are $t_j \in \Lambda$ and integers n_j for j in some finite set of indices J^* , such that*

$$\sum_{j \in J^*} n_j \text{ is odd,} \quad \text{and} \quad \sum_{j \in J^*} n_j t_j = 0.$$

b) *$\text{dist}(\mathbf{x}, \mathbf{y})$ is even, for all $\mathbf{x} \in I$, $\mathbf{y} \in I'$.*

For any $m \geq 0$, there exists a coupling of $\{\tau_e\}_{e \in \mathbb{E}_m^c}$ and $\{\tau'_e\}_{e \in \mathbb{E}_m^c}$ such that if $\{\tau_e\}_{e \in \mathbb{E}_m}$ and $\{\tau'_e\}_{e \in \mathbb{E}_m}$ each have distribution $P_\tau^{\mathbb{E}_m}$, then the marginal distributions of $\{\tau_e\}_{e \in \mathbb{E}}$ and $\{\tau'_e\}_{e \in \mathbb{E}}$ are given by the product measure $P_\tau^{\mathbb{E}}$, and such that if first-passage percolation is performed with $(I, \{\tau_e\}_{e \in \mathbb{E}})$ and $(I', \{\tau'_e\}_{e \in \mathbb{E}})$, respectively, then with probability one there exists an $N_c < \infty$ and a $T_c < \infty$, such that

$$T(v_n) = T'(v_n) \quad \text{and} \quad B_t = B'_t, \quad (6.2)$$

for all $v_n \in \mathbb{V}_{\mathcal{G}_n}$ for $n \geq N_c$, and for all $t \geq T_c$.

Before we construct the couplings, we focus on the promised 0–1 law that follows from Proposition 6.1 and 6.2. For this we will use *Lévy's 0–1 law*. It states that for σ -algebras $\{\mathcal{F}_t\}_{t \geq 0}$ such that $\mathcal{F}_t \uparrow \mathcal{F}_\infty$ as $t \rightarrow \infty$, if $A \in \mathcal{F}_\infty$, then $P(A|\mathcal{F}_t) \rightarrow 1_A$, as $n \rightarrow \infty$, almost surely. A proof for the discrete case can be found in e.g. Durrett (2005, Theorem 4.5.8). The continuous case follows via the Martingale convergence theorem.

Recall that we defined the σ -algebra $\mathcal{T}_t = \sigma(\{B_s\}_{s \geq t})$, and define $\mathcal{F}_t := \sigma(\{B_s\}_{0 \leq s \leq t})$, where as before B_s is the set of infected vertices at time s . We may think of \mathcal{T}_t as the σ -algebra of events $A \in \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$ that do not depend on the times at which vertices were infected before time t . The 0–1 law we shall prove deals with the tail σ -algebra $\mathcal{T} = \bigcap_{t \geq 0} \mathcal{T}_t$.

Theorem 6.3 (0–1 law). *Consider first-passage percolation performed under the assumptions of either Proposition 6.1 or 6.2. Then $P(A) \in \{0, 1\}$, for any event $A \in \mathcal{T}$.*

Note that Theorem 1.9 is a special case of Theorem 6.3.

Proof of Theorem 6.3 from Propositions 6.1 and 6.2. Consider two infections with the respective sets of passage times $\{\tau_e\}_{e \in \mathbb{E}}$ and $\{\tau'_e\}_{e \in \mathbb{E}}$. For $t \geq 0$, let \mathcal{F}_t and \mathcal{F}'_t be σ -algebras generated by their respective realisations up to time t . Let

$$\nu_t = \max \{n \geq 0 : (B_t \cup B'_t) \cap (\mathbb{V}_{\mathcal{G}_n} \cup \mathbb{V}_{\mathcal{G}_{-n}}) \neq \emptyset\}$$

denote the furthest level (in positive or negative direction) infected at time t . Since, almost surely, $\rho_k < \infty$ and $T(\hat{v}_{\rho_k})/k \rightarrow \mu_\tau > 0$ as $k \rightarrow \infty$, then there is a $k = k(t) < \infty$ such that $T(v) > t$ for all $v \in \bigcup_{n \geq \rho_k} \mathbb{V}_{\mathcal{G}_n}$. Thus $\nu_t < \infty$, almost surely, for any $t < \infty$.

For any fixed $t \geq 0$, by Propositions 6.1 and 6.2, there is a coupling of $\{\tau_e\}_{e \in \mathbb{E}_{\nu_t+1}^c}$ and $\{\tau'_e\}_{e \in \mathbb{E}_{\nu_t+1}^c}$, such that there exists an almost surely finite time T_c , such that $B_s = B'_s$ for all $s \geq T_c$. Since $A \in \mathcal{T}_{T_c}$, the outcome of A only depends on B_s for $s \geq T_c$. In particular it has to hold that

$$P(A|\mathcal{F}_t) = P(A|\mathcal{F}'_t).$$

Thus, $P(A|\mathcal{F}_t)$ is nonrandom and equals $P(A)$, for all $t \geq 0$. But, according to Lévy's 0–1 law, $P(A|\mathcal{F}_t) \rightarrow 1_A$ as $t \rightarrow \infty$, almost surely. Hence, $P(A) = 1_A$ almost surely, and therefore $P(A)$ equals either 0 or 1. \square

It remains only to prove Propositions 6.1 and 6.2.

6.1 Exact coupling of time-delayed infections on \mathbb{Z}

Before proving Proposition 6.1 and 6.2, we shall first prove a lemma where we consider two infections on \mathbb{Z} . This lemma will figure as a key step in the proof of Proposition 6.1 and 6.2. For first-passage percolation on \mathbb{Z} , T_n simply takes the form $T_n = \sum_{k=1}^n \tau_k$. If we let the latter infection be delayed for some time T_{delay} , i.e., started at time T_{delay} instead of time zero, then $T'_n = T_{\text{delay}} + \sum_{k=1}^n \tau'_k$. We will construct a coupling of the passage times such that $T_n = T'_n$ for large n . The precise statement is as follows.

Lemma 6.4. *Let T_{delay} be any non-negative constant, and assume that either of the following hold:*

- a) P_τ has an absolutely continuous component (with respect to Lebesgue measure).
- b) P_τ is such that for some finite index set J , there are non-negative integers n_j and n'_j , such that $\sum_{j \in J} n_j = \sum_{j \in J} n'_j$, and for atoms $t_j \in \Lambda$ of P_τ

$$\sum_{j \in J} n_j t_j = \sum_{j \in J} n'_j t_j + T_{\text{delay}}. \quad (6.3)$$

Then, there exists a coupling of $\{\tau_k\}_{k \geq 1}$ and $\{\tau'_k\}_{k \geq 1}$ such that their marginal distributions are that of i.i.d. random variables with distribution P_τ , and such that

$$\sum_{k=1}^n \tau_k = T_{\text{delay}} + \sum_{k=1}^n \tau'_k, \quad \text{for large } n. \quad (6.4)$$

The key to prove this lemma is to (in each case separately) identify a suitable random walk. The identification of the random walk in case *a*) heavily exploits ideas similar to those found in Lindvall (2002, Chapter III.5). In case *b*), a multi dimensional random walk will be based on condition (6.3). This walk is then easily coupled with known techniques found e.g. in Lindvall (2002, Chapter II.12–17).

Proof of case a). Let $[a, b]$ be an interval on which P_τ has density $\geq c$, for some $c > 0$. Define

$$\delta := \max \left\{ d \geq 0 : d \leq \frac{b-a}{2}, d = \frac{T_{\text{delay}}}{m} \text{ for some } m \in \mathbb{N} \right\}.$$

Couple $\{\tau_k\}_{k \geq 1}$ and $\{\tau'_k\}_{k \geq 1}$ in the following way. With probability $1 - c2\delta$ we choose $\tau_k = \tau'_k$, drawn from the distribution

$$\tilde{P}_\tau(\cdot) := (P_\tau(\cdot) - c\lambda(\cdot \cap [a, a + 2\delta])) / (1 - c2\delta),$$

where λ denotes Lebesgue measure. With the remaining probability $c2\delta$, draw τ_k uniformly on the interval $[a, a + 2\delta]$, and choose τ'_k as

$$\tau'_k = \begin{cases} \tau_k + \delta, & \text{if } \tau_k \leq a + \delta \\ \tau_k - \delta, & \text{if } \tau_k > a + \delta \end{cases}.$$

That τ'_k also is uniformly distributed on $[a, a + 2\delta]$ is immediate. Thus, it is easy to see that the marginal distribution of both τ_k and τ'_k is P_τ , and this is indeed a coupling of the two infections.

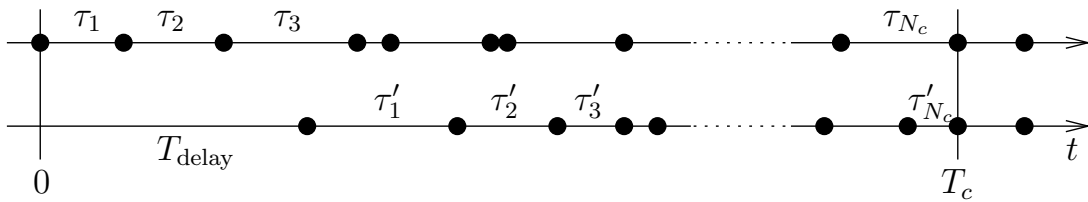


Figure 5: The dots represent the times at which the respective infection spreads. In this realisation $\tau_1 = \tau'_1 - \delta$, $\tau_2 = \tau'_2$ and $\tau_3 = \tau'_3 + \delta$. The coupling is constructed such that after some time T_c , both infections reach some level N_c simultaneously.

The coupling is such that each time τ_k and τ'_k are chosen differently, the difference $\{D_n\}_{n \geq 1}$, where $D_n := T_{\text{delay}} + \sum_{k=1}^n (\tau'_k - \tau_k)$ will jump $\pm \delta$. Since $T_{\text{delay}} = m\delta$, for some integer m , D_n constitutes a simple random walk on $\delta\mathbb{Z}$. Let N_c denote the first n for which D_n hits zero. From

this moment on, τ_k and τ'_k are chosen identically, and (6.4) holds for $n \geq N_c$. That the coupling is successful is easily seen, since

$$P(N_c < \infty) = P(\exists n : D_n = 0) \geq P(\exists n : D_n = 0 | \tau_k \neq \tau'_k \text{ i.o.}) P(\tau_k \neq \tau'_k \text{ i.o.}) = 1,$$

where 'i.o.' abbreviates 'infinitely often'. The last equality follows from the recurrence of a 1-dimensional simple random walk, and Borel-Cantelli's second lemma.

Proof of case b). By assumption, for some set $\{t_j\}_{j \in J} \subseteq \Lambda$ of atoms for the distribution P_τ , there are non-negative integers n_j and n'_j such that $\sum_{j \in J} n_j = \sum_{j \in J} n'_j$ and (6.3) holds.

It is easily seen that we may assume that J , n_j and n'_j are chosen such that for each $j \in J$, exactly one of the integers n_j and n'_j is positive. We introduce integer valued random variables

$$\begin{aligned} X_j^n &= \#\{k \leq n : \tau_k = t_j\} - n_j, \\ Y_j^n &= \#\{k \leq n : \tau'_k = t_j\} - n'_j. \end{aligned}$$

Define $Z_j^n = X_j^n - Y_j^n$. It is clear that we from (6.3) can conclude that (6.4) holds, if $Z_j^n = 0$ for all $j \in J$ and $\tau_k = \tau'_k$ for all $k \leq n$ such that $\tau_k \notin \{t_j\}_{j \in J}$ or $\tau'_k \notin \{t_j\}_{j \in J}$.

Let $J_n = \{j \in J : Z_j^n \neq 0\}$, let $p_j = P_\tau(t_j)$, and $q_n = \sum_{j \in J_n} p_j$. In particular, $J_0 = J$. Couple $\{\tau_k\}_{k \geq 1}$ and $\{\tau'_k\}_{k \geq 1}$ by choosing τ_k and τ'_k identically from the distribution

$$\tilde{P}_\tau(\cdot) := \frac{1}{1 - q_{k-1}} \left(P_\tau(\cdot) - \sum_{j \in J_{k-1}} p_j \mathbf{1}_{\{t_j\}}(\cdot) \right)$$

with probability $1 - q_{k-1}$. With remaining probability q_{k-1} we choose τ_k and τ'_k independently with distribution $P(\tau = t_j) = \frac{p_j}{q_{k-1}}$, for $j \in J_{k-1}$. The marginal distribution of τ_k and τ'_k is seen to be P_τ , whence this is a coupling of $\{\tau_k\}_{k \geq 1}$ and $\{\tau'_k\}_{k \geq 1}$.

Note that $\tau_k = \tau'_k$ for all k such that $\tau_k \notin \{t_j\}_{j \in J}$ and $\tau'_k \notin \{t_j\}_{j \in J}$. For each fixed $j \in J$, $\{Z_j^n\}_{n \geq 0}$ will, as n increases, jump ± 1 with equal probability. Hence, for fixed j , $\{Z_j^n\}_{n \geq 0}$ constitutes a simple random walk on \mathbb{Z} . Note that if n^* denotes the first n such that $Z_j^n = 0$, then, by definition, $j \in J_n$ for $n < n^*$, but $j \notin J_n$ for $n \geq n^*$.

By assumption we have that

$$\sum_{j \in J} Z_j^0 = \sum_{j \in J} (n_j - n'_j) = 0.$$

Moreover, the sum of Z_j^n is constant for all n , i.e.,

$$\sum_{j \in J} Z_j^n = \sum_{j \in J} Z_j^0 = 0.$$

It follows that it is not possible for $|J_n| = 1$ for some n . There will therefore always be a positive probability to choose $\tau_{n+1} \neq \tau'_{n+1}$ as long as Z_j^n for some j . From this observation, Borel-Cantelli's second lemma and the recurrence of 1-dimensional simple random walks, we may further conclude that $P(\exists n : Z_j^n = 0) = 1$ for each $j \in J$. Let $N_c = \min\{n \geq 0 : J_n = \emptyset\}$.

For $n \geq N_c$ we have $Z_j^n = 0$ for all $j \in J$, and (6.4) holds for every such n . The coupling is successful since

$$P(N_c < \infty) = P\left(\bigcap_{j \in J} \{\exists n : Z_j^n = 0\}\right) = 1. \quad \square$$

6.2 Exact coupling of two infections

In order to prove Proposition 6.1 and 6.2, we will arrange matters so that Lemma 6.4 can be applied. First, we need some notation. Recall from Section 2.2 that E_n denotes the set of edges between level n and $n + 2M$, including edges at level n and level $n + 2M$. In (2.2) we defined $\hat{E}_n = \gamma_n \cup \mathbb{E}_{\mathcal{G}_n} \cup \mathbb{E}_{\mathcal{G}_{n+2M}}$, where γ_n is a path of shortest length between $\mathbb{V}_{\mathcal{G}_n}$ and $\mathbb{V}_{\mathcal{G}_{n+2M}}$. Introduce the notation \hat{e}_{n+M} for the edge in γ_n with endpoints \hat{v}_{n+M} and u , where \hat{v}_{n+M} is the vertex in $\mathbb{V}_{\mathcal{G}_{n+M}}$ first reached by γ_n , and u the vertex first reached after \hat{v}_{n+M} by γ_n . Define the event

$$A_n^* := \left\{ \tau_e \leq t', \forall e \in \hat{E}_n \setminus \{\hat{e}_{n+M}\} \right\} \cap \left\{ \tau_e \geq t'', \forall e \in E_n \setminus \hat{E}_n \right\}.$$

Note that $A_n = A_n^* \cap \{\tau_{\hat{e}_{n+M}} \leq t'\}$ for A_n as defined in (2.4).

We will next prove Proposition 6.1, which is a slightly stronger version of Proposition 1.10. We first outline the general idea. It follows from the regenerative behaviour that if $\tau_e = \tau'_e$ for all $e \in \mathbb{E}$, then there is a real number T_d such that

$$B_t \cap \bigcup_{n \geq 0} \mathbb{V}_{\mathcal{G}_n} = B'_{t+T_d} \cap \bigcup_{n \geq 0} \mathbb{V}_{\mathcal{G}_n} \quad (6.5)$$

for t large enough. The idea for the coupling is to assign identical passage times for both infections, that is $\tau_e = \tau'_e$, except for certain edges which we make sure both infections have to pass. More precisely, for some sequence $\{l_k\}_{k \geq 0}$, for k such that $A_{l_k}^*$ occurs, choose either the passage times for \hat{e}_{l_k+M} independently at most t' , or equal. This generates a sequence of edges for which we invoke Lemma 6.4. That is, we make sure that $\{T(\hat{v}_{\rho_n}) - T'(\hat{v}_{\rho_n})\}_{n \geq 1}$ performs a random walk which eventually hits zero. This implies that (6.5) holds, with $T_d = 0$, for t large enough. This will complete the coupling of the infections in the direction of increasing levels. The opposite direction is treated in the same way.

Proof of Proposition 6.1. By assumption, P_τ has an absolutely continuous component, so suppose that $[a, b]$ is an interval on which P_τ has density $\geq c > 0$. Let $a < t' < t'' < b$ and choose M in accordance with Lemma 2.3. We may further assume that $I \cup I'$ contains no vertex beyond level m . Let $l_k := m + k(2M + 1)$ for integers $k \geq 0$. Couple $\{\tau_e\}_{e \in \mathbb{E}_m^c}$ and $\{\tau'_e\}_{e \in \mathbb{E}_m^c}$ by choosing $\tau_e = \tau'_e$ with distribution P_τ , independently for all e at level m or beyond such that $e \neq \hat{e}_{l_k+M}$ for some $k \geq 0$. Independently for $k \geq 0$, let

$$(\xi_k, \xi'_k) = \begin{cases} (\theta_k, \theta'_k), & \text{with probability } P_\tau([0, t']) \\ (\eta_k, \eta_k), & \text{with probability } 1 - P_\tau([0, t']), \end{cases}$$

where θ_k and θ'_k are to be coupled below, so that they both have marginal distribution $P_\tau(\cdot \mid \tau \leq t')$, and η_k has distribution $P_\tau(\cdot \mid \tau > t')$. For the set of edges $\{\hat{e}_{l_k+M}, \text{ for } k \geq 0\}$, we choose

the pair

$$\left(\tau_{\hat{e}_{l_k+M}}, \tau'_{\hat{e}_{l_k+M}}\right) = \begin{cases} (\xi_k, \xi'_k), & \text{if } A_{l_k}^* \text{ occurs} \\ (\tau_k, \tau_k), & \text{otherwise,} \end{cases}$$

where τ_k is distributed according to P_τ , independently for all k . One realises from the coupling that the marginal distributions of both τ_e and τ'_e is P_τ , for every edge e .

Note that the only edges for which τ_e and τ'_e may differ, are the edges \hat{e}_{l_k+M} for $k \geq 0$ such that A_{l_k} occurs. Let κ_j denote the index k for which A_{l_k} occurs for the j th time. That

$$\left(\tau_{\hat{e}_{l_{\kappa_j}+M}}, \tau'_{\hat{e}_{l_{\kappa_j}+M}}\right) = (\theta_{\kappa_j}, \theta'_{\kappa_j}) \quad (6.6)$$

is equivalent to that $A_{l_{\kappa_j}}$ occurs. Since $P(A_{l_k}) > 0$, we will have an infinite sequence $\{\kappa_j\}_{j \geq 1}$ such that (6.6) holds. We now claim that the proposition will follow if we apply Lemma 6.4 to the sequences $\{\theta_{\kappa_j}\}_{j \geq 1}$ and $\{\theta'_{\kappa_j}\}_{j \geq 1}$, with distribution $P_\tau(\cdot | \tau \leq t')$, and

$$T_{\text{delay}} = \left| T(\hat{v}_{l_{\kappa_1}+M}) - T'(\hat{v}_{l_{\kappa_1}+M}) \right|.$$

To see this, we use Lemma 2.3. Given A_{l_k} , the path along which any vertex at level $l_k + 2M$ or beyond is infected has to pass the edge \hat{e}_{l_k+M} . By the coupling $\tau_e = \tau'_e$ for all e at level l_{κ_1} or beyond such that $e \neq \hat{e}_{l_{\kappa_j}+M}$ for $j \geq 1$. Moreover, $\tau_e = \theta \leq t'$ and $\tau'_e = \theta' \leq t'$ for $e \in \{\hat{e}_{l_{\kappa_j}+M}, \text{ for } j \geq 1\}$. It follows that each vertex at level $l_{\kappa_1} + 2M + 1$ and beyond, will be reached in the same order. Since P_τ is absolutely continuous on $[a, b]$ and $t' > a$, $P_\tau(\cdot | \tau \leq t')$ is absolutely continuous on $[a, t']$. Condition a) of Lemma 6.4 is therefore fulfilled. Coupling $\{\theta_{\kappa_j}\}_{j \geq 1}$ and $\{\theta'_{\kappa_j}\}_{j \geq 1}$ according to the lemma we will have with probability one that, from some level on, both infections will reach each vertex at the same time, i.e.,

$$T(v_n) = T'(v_n) \quad (6.7)$$

for any $v_n \in \mathbb{V}_{\mathcal{G}_n}$, for n sufficiently large.

The infections may in the same manner be coupled along the negative coordinate axis. Doing this, then there is $N_c \in \mathbb{N}$ such that (6.7) holds for $|n| \geq N_c$. In almost surely finite time, each vertex at level n , for $|n| \leq N_c$, will be infected. Hence, we conclude that for some almost surely finite time T_c ,

$$B_t = B'_t, \quad \text{for each } t \geq T_c. \quad \square$$

In preparation for the proof of Proposition 6.2, we restrict our attention to (K, d) -tubes. Let F_n denote the set of edges between level n and $n + 2M + 4\beta$, for integers

$$M > \frac{(d-1)(K-1)t'}{t'' - t'} \quad \text{and} \quad \beta > \frac{t'}{t'' - t'}.$$

Let $\mathbf{e}_1 = (1, 0, \dots, 0)$. Denote by $e_{u,n}$ the edge between $(n+M+\beta)\mathbf{e}_1$ and $(n+M+\beta, 1, 0, \dots, 0)$, and by $e_{d,n}$ the edge between $(n+M+3\beta)\mathbf{e}_1$ and $(n+M+3\beta, 1, 0, \dots, 0)$. Let γ_n^* denote the path of shortest length from $\mathbf{n} = n\mathbf{e}_1$ to $(n+2M+4\beta)\mathbf{e}_1$. Let γ_n^{**} denote the path of shortest

length from \mathbf{n} to $(n + 2M + 4\beta)\mathbf{e}_1$ that visits the four endpoints of $e_{u,n}$ and $e_{d,n}$. Let \hat{F}_n and \hat{H}_n be defined as (see Figure 6)

$$\begin{aligned}\hat{F}_n &:= \gamma_n^* \cup \{e_{u,n}, e_{d,n}\} \cup \mathbb{E}_{\mathcal{G}_n} \cup \mathbb{E}_{\mathcal{G}_{n+2M+4\beta}} \\ \hat{H}_n &:= \gamma_n^{**} \cup \mathbb{E}_{\mathcal{G}_n} \cup \mathbb{E}_{\mathcal{G}_{n+2M+4\beta}}.\end{aligned}$$

For constants t' and t'' such that $m_\tau < t' < t'' < M_\tau$, define the events

$$\begin{aligned}C_n &:= \left\{ \tau_e \leq t', \forall e \in \hat{F}_n \right\} \cap \left\{ \tau_e \geq t'', \forall e \in F_n \setminus \hat{F}_n \right\}, \\ D_n &:= \left\{ \tau_e \leq t', \forall e \in \hat{H}_n \right\} \cap \left\{ \tau_e \geq t'', \forall e \in F_n \setminus \hat{H}_n \right\}.\end{aligned}$$

Trivially $P(C_n) = P(D_n) > 0$, since \hat{F}_n and \hat{H}_n contain equally many edges.

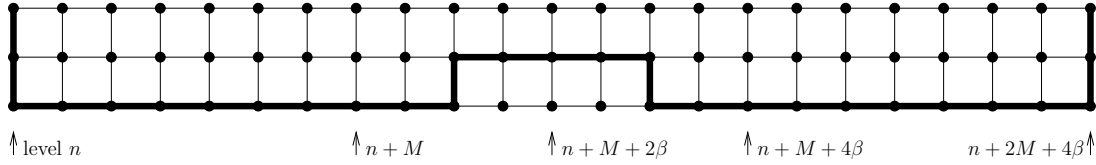


Figure 6: The $(3, 2)$ -tube between level n and $n + 2M + 4\beta$. If D_n occurs, the infection will advance along the thick edges.

Recall that $\rho_I = \max\{n \in \mathbb{Z} : \mathbb{V}_{\mathcal{G}_n} \cap I \neq \emptyset\}$. The following lemma says that given that the event C_n (or D_n) occurs, the infection will in order to reach level $n + 2M + 4\beta$ from level n do so via γ_n^* (or γ_n^{**}).

Lemma 6.5. *Let t' and t'' be constants such that $m_\tau < t' < t'' < M_\tau$, and assume that $n \geq \rho_I$. Given C_n (respectively D_n), then*

$$T(v) = T(\mathbf{n}) + T(\Gamma) + T((n + 2M + 4\beta)\mathbf{e}_1, v),$$

for each v at level $n + 2M + 4\beta$ or beyond, where $\Gamma = \gamma_n^*$ (respectively $\Gamma = \gamma_n^{**}$).

It is easy to see that the infection, from level n to level $n + 2M + 4\beta$, inevitably has to follow the paths γ_n^* and γ_n^{**} , in their respective cases, reasoning in a similar way as in the proof of Lemma 2.3. We leave the details to the reader.

The coupling of Proposition 6.2 will be constructed in two steps. The second part is similar to the coupling in the proof of Proposition 6.1. The first part is needed to make sure that condition b) of Lemma 6.4 will be satisfied. Before we give the somewhat technical proof, we present the idea behind the first step.

The events C_n and D_n were defined with respect to passage times from the sequence $\{\tau_e\}_{e \in \mathbb{E}}$. Let C'_n and D'_n denote the analogous events with respect to the sequence $\{\tau'_e\}_{e \in \mathbb{E}}$. Assign identical passage times for both infections, except for some edges in F_{l_k} , for some sequence $\{l_k\}_{k \geq 0}$. The remaining edges we couple in order to make the event C_{l_k} occur simultaneously as D'_{l_k} , and D_{l_k} occur simultaneously as C'_{l_k} . When they happen, the difference in length of the

minimising paths in $T_{n+2M+4\beta}$ and $T'_{n+2M+4\beta}$ will either increase or decrease by 2. Thus, the difference in length constitutes a random walk. End the first step when it hits either 0 or the odd number $\omega = \sum_{j \in J^*} n_j$, for $\{n_j\}_{j \in J^*}$ as in assumption a) of Proposition 6.2. We will see that condition b) of Lemma 6.4 is then satisfied for $T_{\text{delay}} = |T'(v) - T(v)|$, for some vertex v .

Proof of Proposition 6.2. We may assume that $I \cup I'$ contains no vertex beyond level m . Set $l_k := m + k(2M + 4\beta + 1)$ for $k \geq 0$. For $j = 1, 2, \dots, 2\beta$, let $f_{k,j}$ (and $h_{k,j}$) denote the edge in \hat{F}_{l_k} (and \hat{H}_{l_k}) between level $l_k + M + \beta + j - 1$ and $l_k + M + \beta + j$ (respectively).

Couple $\{\tau_e\}_{e \in \mathbb{E}_m^c}$ and $\{\tau'_e\}_{e \in \mathbb{E}_m^c}$ in the following way. For one k at the time, choose $\tau_e = \tau'_e$ with distribution P_τ , independently for every edge e between level l_k and l_{k+1} , not at level l_{k+1} nor among $\{f_{k,j}, h_{k,j} : j = 1, 2, \dots, 2\beta\}$. For $j = 1, 2, \dots, 2\beta$, choose $\tau_{f_{k,j}}$ and $\tau_{h_{k,j}}$ independently with distribution P_τ , and set

$$\left(\tau'_{f_{k,j}}, \tau'_{h_{k,j}} \right) = \begin{cases} (\tau_{h_{k,j}}, \tau_{f_{k,j}}), & \text{if } C_{l_k} \cup D_{l_k} \\ (\tau_{f_{k,j}}, \tau_{h_{k,j}}), & \text{otherwise.} \end{cases}$$

Trivially τ_e has distribution P_τ , and it is easy to see that the marginal distribution of τ'_e for each e also is P_τ . Note that the coupling is such that C'_{l_k} occurs if and only if D_{l_k} occurs. In addition, D'_{l_k} occurs if and only if C_{l_k} does.

Let z_k , for $k \geq 1$, denote the length of the path of shortest passage time from I to level $l_k + 2M + 4\beta$, with respect to $\{\tau_e\}_{e \in \mathbb{E}}$. When several paths are possible, choose one. Similarly, let z'_k denote the length of the path of shortest passage time from I' to level $l_k + 2M + 4\beta$, with respect to $\{\tau'_e\}_{e \in \mathbb{E}}$. When several paths are possible, choose one that minimises $|z_k - z'_k|$. Set $\zeta_k := z_k - z'_k$.

With help from Lemma 6.5, we draw the following conclusions. For each k such that C_{l_k} (and therefore also D'_{l_k}) occurs, $\zeta_k - \zeta_{k-1} = -2$. When D_{l_k} (and therefore also C'_{l_k}) occurs, $\zeta_k - \zeta_{k-1} = 2$. Otherwise $\zeta_k = \zeta_{k-1}$. Thus, $\{\zeta_k\}_{k \geq 1}$ constitutes a simple random walk on either $2\mathbb{Z}$ or $2\mathbb{Z} + 1$, depending on the value of ζ_1 . Such walk is recurrent and will with probability one, reach either zero or the odd number $\omega := \sum_{j \in J^*} n_j$, respectively. Let κ denote the first k for which this happens. Couple the infections along the negative coordinate axis in the same way.

The first part of the coupling is done, and before we continue with the second part, we shall verify that assumption b) of Lemma 6.4 is satisfied. We may assume that $T_{l_\kappa+2M+4\beta} \leq T'_{l_\kappa+2M+4\beta}$. Set

$$T_{\text{delay}} = T'_{l_\kappa+2M+4\beta} - T_{l_\kappa+2M+4\beta}.$$

Given $\{(\tau_e, \tau'_e)\}_{e \in \mathbb{E}_{l_\kappa+2M+4\beta}}$, we may represent the passage time for each infection as

$$T_{l_\kappa+2M+4\beta} = \sum_{j \in J} m_j t_j \quad \text{and} \quad T'_{l_\kappa+2M+4\beta} = \sum_{j \in J'} m'_j t_j,$$

for index sets J and J' , $t_j \in \Lambda$, and positive integers m_j and m'_j that indicate the number of edges e in the minimising path to level $l_\kappa + 2M + 4\beta$ such that $\tau_e = t_j$ and $\tau'_e = t_j$, respectively.

If $\zeta_\kappa = 0$, then $\sum_{j \in J} m_j = \sum_{j \in J'} m'_j$, and assumption b) of Lemma 6.4 is directly satisfied, since

$$T_{\text{delay}} + T_{l_\kappa+2M+4\beta} = T'_{l_\kappa+2M+4\beta}.$$

Note that this will be the case if $\text{dist}(\mathbf{x}, \mathbf{y})$ is even, for all $\mathbf{x} \in I$, $\mathbf{y} \in I'$, since then $\zeta_k \in 2\mathbb{Z}$. If rather $\zeta_k = \omega$ is odd, we need the additional assumption that $\sum_{j \in J^*} n_j t_j = 0$ for some index set J^* , point masses t_j , and integers n_j such that $\sum_{j \in J^*} n_j = \omega$. Then, assumption b) of Lemma 6.4 is again satisfied, since

$$T_{\text{delay}} + T_{l_\kappa + 2M + 4\beta} = T'_{l_\kappa + 2M + 4\beta} + \sum_{j \in J^*} n_j t_j.$$

We will now go on with the second part of the coupling. Let

$$t^* := \max\{t_j \in \Lambda : j \in J \cup J' \cup J^*\}.$$

It may be the case that $t^* = M_\tau$ as defined in (2.3). This makes it necessary to introduce some extra notation. Write E'_n for the set of edges between level n and $n + 2M + 1$, and let γ'_n denote the path of shortest length from \mathbf{n} to $(n + 2M + 1)\mathbf{e}_1$. Let

$$\hat{E}'_n = \gamma'_n \cup \mathbb{E}_{\mathcal{G}_n} \cup \mathbb{E}_{\mathcal{G}_{n+2M+1}}.$$

Denote by \hat{e}_n the edge between \mathbf{n} and $(n + 1)\mathbf{e}_1$. Let X_n denote the set of edges connecting a vertex at level n with one at level $n + 1$, excluding the edge \hat{e}_n . Define the event

$$\begin{aligned} A_n^{**} = & \left\{ \tau_e \leq t', \forall e \in \hat{E}'_n \setminus \{\hat{e}_{n+M}\} \right\} \cap \left\{ \tau_e \geq t^*, \forall e \in X_{n+M} \right\} \\ & \cap \left\{ \tau_e \geq t'', \forall e \in E'_n \setminus (\hat{E}'_n \cup X_n) \right\}. \end{aligned}$$

Let $\lambda_k := l_{\kappa+1} + k(2M + 2)$ for $k \geq 0$. Continue the coupling of $\{\tau_e\}_{e \in \mathbb{E}_m^c}$ and $\{\tau'_e\}_{e \in \mathbb{E}_m^c}$ by choosing $\tau_e = \tau'_e$ with distribution P_τ , independently for all e at level λ_0 or beyond such that $e \neq \hat{e}_{\lambda_k+M}$ for some $k \geq 0$. Independently for $k \geq 0$, let

$$(\xi_k, \xi'_k) = \begin{cases} (\theta_k, \theta'_k), & \text{with probability } P_\tau([0, t^*]) \\ (\eta_k, \eta_k), & \text{with probability } 1 - P_\tau([0, t^*]), \end{cases}$$

where θ_k and θ'_k have marginal distribution $P_\tau(\cdot | \tau \leq t^*)$, and η_k has distribution $P_\tau(\cdot | \tau > t^*)$ (η_k is not needed when $t^* = M_\tau$). For the set of edges $\{\hat{e}_{\lambda_k+M}, \text{ for } k \geq 0\}$ we couple their passage times as

$$(\tau_{\hat{e}_{\lambda_k+M}}, \tau'_{\hat{e}_{\lambda_k+M}}) = \begin{cases} (\xi_k, \xi'_k), & \text{if } A_{\lambda_k}^{**} \text{ occurs} \\ (\tau_k, \tau_k), & \text{otherwise,} \end{cases}$$

where τ_k is distributed according to P_τ , independently for all k . One realises from the coupling that the marginal distributions of both τ_e and τ'_e is P_τ .

Note that the only edges for which τ_e and τ'_e may differ, are the edges \hat{e}_{λ_k+M} for $k \geq 0$ such that $A_{\lambda_k}^{**} \cap \{\tau_{\hat{e}_{\lambda_k+M}} \leq t^*\}$ occurs. Let κ_j denote the index k for which $A_{\lambda_k}^{**} \cap \{\tau_{\hat{e}_{\lambda_k+M}} \leq t^*\}$ occurs for the j th time. That

$$(\tau_{\hat{e}_{\lambda_{\kappa_j}+M}}, \tau'_{\hat{e}_{\lambda_{\kappa_j}+M}}) = (\theta_{\kappa_j}, \theta'_{\kappa_j}) \tag{6.8}$$

is equivalent to that $A_{\lambda_k}^{**} \cap \{\tau_{\hat{e}_{\lambda_k+M}} \leq t^*\}$ occurs. Since $P(A_{\lambda_k}^{**} \cap \{\tau_{\hat{e}_{\lambda_k+M}} \leq t^*\}) > 0$, we will have an infinite sequence $\{\kappa_j\}_{j \geq 1}$ such that (6.8) holds. We now claim that the proposition will follow if we apply Lemma 6.4 to the sequences $\{\theta_{\kappa_j}\}_{j \geq 1}$ and $\{\theta'_{\kappa_j}\}_{j \geq 1}$, with distribution $P_\tau(\cdot | \tau \leq t^*)$ and T_{delay} as defined above.

To see this, argue as in the proof of Lemma 2.3. Given $A_{\lambda_k}^{**} \cap \{\tau_{\hat{e}_{\lambda_k+M}} \leq t^*\}$, the path along which any vertex at level $\lambda_k + 2M + 1$ or beyond is infected inevitably has to pass the edge \hat{e}_{λ_k+M} . By the coupling, $\tau_e = \tau'_e$ for all e at level λ_{κ_1} or beyond such that $e \neq \hat{e}_{\lambda_{\kappa_j}+M}$ for some $j \geq 1$. Moreover, $\tau_e = \theta \leq t'$ and $\tau'_e = \theta' \leq t'$ for $e \in \{\hat{e}_{\lambda_{\kappa_j}+M} \text{ for } j \geq 1\}$. Therefore, each vertex at level $\lambda_{\kappa_1} + 2M + 1$ and beyond, will be reached in the same order for both infections. Coupling $\{\theta_{\kappa_j}\}_{j \geq 1}$ and $\{\theta'_{\kappa_j}\}_{j \geq 1}$ according to Lemma 6.4 we will have with probability one that, from some level on, both infections will reach each vertex at the same time, i.e.,

$$T(v_n) = T'(v_n) \quad (6.9)$$

for any $v_n \in \mathbb{V}_{\mathcal{G}_n}$ for n sufficiently large. Since we chosen t^* as large as we did, we made sure that $P_\tau(\cdot | \tau \leq t^*)$ meets assumption b) of Lemma 6.4.

The infections may in the same manner be coupled along the negative coordinate axis. Doing this, then there is $N_c \in \mathbb{N}$ such that (6.7) holds for $|n| \geq N_c$. In almost surely finite time, each vertex at level n , for $|n| \leq N_c$, will be infected. Hence, we conclude that for some almost surely finite time T_c ,

$$B_t = B'_t, \quad \text{for each } t \geq T_c. \quad \square$$

Remark 6.6. There exists in general no exact coupling of two infections with discrete passage time distribution on arbitrary 1-dimensional periodic graphs. Consider the distribution $P_\tau(1) = P_\tau(1 + 3/5) = 1/2$. P_τ satisfies the assumption of Proposition 6.2, whence there is an exact coupling of two infections on the (K, d) -tube, for $K, d \geq 2$.

Consider instead the graph with set of vertices $\mathbb{Z} \times \{0, 1\}$ and where two vertices are connected by an edge if their Euclidean distance is $\leq \sqrt{2}$. Note that with the above passage time distribution, in order to reach any vertex at level n , an infection will always pass exactly n edges. This is easily seen by realising that no vertical edge will ever be used in order to reach an uninfected vertex. Thus, for two infections started with $I = \{(0, 0)\}$ and $I' = \{(m, 0)\}$, we will have

$$\begin{aligned} |T(\mathbf{n}) - T'(\mathbf{n})| &\geq \inf_{\substack{a+b=n \\ a'+b'=n-m}} \left| a - a' + (b - b') \left(1 + \frac{3}{5}\right) \right| \\ &= \inf_{b-b' \in \mathbb{Z}} \left| m - \frac{3(b-b')}{5} \right| \geq \frac{1}{5}, \end{aligned}$$

for any m that is not a multiple of 3. As we can see, an exact coupling is not possible. \square

Remark 6.7. Condition a) of Proposition 6.2 is due to the fact that the (K, d) -tube is bipartite, i.e., that every circuit has even length. As seen in Remark 6.6, not every non-bipartite graph has an exact coupling without condition a). But, condition a) and b) of Proposition 6.2 could be dropped for e.g. the class of triangular graphs with vertex set $\mathbb{Z} \times \{0, 1, \dots, K-1\}$ and where two vertices at Euclidean distance is 1 and every two vertices (n, m) and $(n+1, m+1)$ for any $n \in \mathbb{Z}$ and $m = 0, 1, \dots, K-2$, are connected by an edge. The necessary modifications of the first part of the proof, and of the event D_n in particular, are easily made. \square

Remark 6.8. If $\text{dist}(\mathbf{x}, \mathbf{y})$ is odd, for all $\mathbf{x} \in I$, $\mathbf{y} \in I'$, then condition a) of Proposition 6.2 is necessary. To see this, assume that an exact coupling is possible. In particular, $T(v) = T'(v)$ for some vertex v . But, if one infection has an even number of edges to pass in order to reach v , the other has an odd number of edges to pass. Thus,

$$0 = T(v) - T'(v) = \sum_{j \in J} n_j t_j - \sum_{j \in J} n'_j t_j,$$

for integers n_j and n'_j such that $\sum_{j \in J} (n_j - n'_j)$ is odd. Hence, condition a) holds. \square

Remark 6.9. Condition a) of Lemma 6.4 can be weakened to distributions P_τ whose convolution with itself has an absolutely continuous component. In fact, it is sufficient if P_τ convoluted with itself n times, for some $n \geq 0$, has an absolutely continuous component. Since the distribution of a sum of independent random variables is the convolution of the individual distributions, we may instead of specifying how to choose (τ_j, τ'_j) for $j \geq 1$, choose $(\sum_{k=(j-1)n+1}^{jn} \tau_k, \sum_{k=(j-1)n+1}^{jn} \tau'_k)$ according to the same specification. Consequently, the assumption on P_τ of Proposition 6.1 can be weakened to involve distributions whose convolution with itself n times has an absolutely continuous component. The modifications are left to the reader.

An example of a distribution that does not have an absolutely continuous component, but whose convolution does, is given by the following. Let ξ_0, ξ_1, \dots be i.i.d. Bernoulli(1/2)-distributed random variables. Define τ to have binary expansion

$$\tau := \begin{cases} (0, \xi_1, 0, \xi_3, 0, \dots), & \text{with probability } \frac{1}{2} \\ (\xi_0, 0, \xi_2, 0, \xi_4, \dots), & \text{otherwise.} \end{cases}$$

Let τ_1 and τ_2 be two independent random variables distributed as τ , and let A denote the event that one of τ_1 and τ_2 has all even coordinates equal to zero and the other has all odd coordinates equal to zero. Neither τ_1 nor τ_2 is absolutely continuous, but the conditional distribution of $\tau_1 + \tau_2$ given A is uniformly distributed on $[0, 1]$. Hence the distribution of $\tau_1 + \tau_2$ has an absolutely continuous component. \square

6.3 No exact coupling possible on trees

We have seen that there is an exact coupling of two first-passage percolation infections on any essentially 1-dimensional periodic graph when the passage time distribution has an absolutely continuous component. We also saw how this sort of coupling gave rise to a 0–1 law. One may ask whether a continuous component is sufficient for an analogous coupling, and corresponding 0–1 law, on any graph? We will answer this question no, by showing that the binary tree \mathbb{T}^2 constitutes a counterexample. \mathbb{T}^2 is the infinite graph that does not contain any circuit, and where each vertex has three neighbours. The graph is completely homogeneous and one vertex, called the *root*, is chosen for reference. Let $\{\tau_e\}_{e \in \mathbb{E}}$ be a set of independent and exponentially distributed passage times associated with the edge set \mathbb{E} of \mathbb{T}^2 , and analogous to before, let

$$B_t = \{v \in \mathbb{V} : T(\text{root}, v) \leq t\}.$$

The following argument is based on the theory of continuous branching processes. Define the front line of the infection at time t as

$$F_t := \#\{v \notin B_t : v \text{ shares an edge with some } u \in B_t\}.$$

Note that $F_0 = 3$ and that F_t increases by one, when B_t does. Hence, F_t can be seen as a continuous time branching process with F_t individuals at time t . Each individual gives with probability one birth to two children (and dies) after an exponentially distributed time, independent of one another. It is well-known (see e.g. Athreya and Ney (1972, Theorems III.7.1–2)) that, for some Malthusian parameter $\lambda > 0$,

$$\exists W := \lim_{t \rightarrow \infty} F_t e^{-\lambda t}, \quad \text{almost surely,} \quad (6.10)$$

and that $E[W] = 3$. Let τ_{e_1} , τ_{e_2} and τ_{e_3} denote the passage time of the edges connected to the root, and let \tilde{F}_t denote F_t conditioned on $\{\tau_{e_1}, \tau_{e_2}, \tau_{e_3} \geq 1\}$. Then, by the lack-of-memory property of the exponential distribution, we have that $\tilde{F}_{t+1} \stackrel{d}{=} F_t$ for any $t \geq 0$. Thus, by (6.10) we have almost surely

$$\lim_{t \rightarrow \infty} \tilde{F}_t e^{-\lambda t} \stackrel{d}{=} e^{-\lambda} \lim_{t \rightarrow \infty} F_t e^{-\lambda t} = e^{-\lambda} W,$$

and we conclude that W is almost surely non-constant. Note that the event

$$\{W = \lim_{t \rightarrow \infty} F_t e^{-\lambda t} \leq x\} \in \mathcal{T}, \quad \text{for every } x.$$

Then, a 0–1 law analogous to Theorem 6.3 cannot hold for first-passage percolation on \mathbb{T}^2 , since this would imply that $P(W \leq x) \in \{0, 1\}$, i.e., that W is almost surely constant.

Acknowledgement: The author is grateful to his supervisor Olle Häggström for introducing him to this problem, as well as for his valuable advices along the way. He would also like to thank Erik Broman for his constructive remarks on the manuscript, Vladas Sidoravicius for interesting discussions, as well as Andreas Nordvall-Lagerås for pointing out the book by Gut (2009), which enriched an earlier version of this paper.

References

- D. Ahlberg. Asymptotics of first-passage percolation on 1-dimensional graphs. Licentiate thesis, available at <http://www.math.chalmers.se/Math/Research/Preprints/2008/39.pdf>, 2008.
- D. Ahlberg. The asymptotic shape, large deviations and dynamical stability in first-passage percolation on cones. Available at <http://www.math.chalmers.se/~md1ahlbda/>, 2011.
- K. B. Athreya and P.E. Ney. *Branching processes*. Springer, Berlin, 1972.
- M. Benaïm and R. Rossignol. A modified Poincaré inequality and its application to first passage percolation. Available at <http://arxiv.org/abs/math.PR/0602496>, 2006.
- M. Benaïm and R. Rossignol. Exponential concentration for first passage percolation through modified Poincaré inequalities. *Ann. Inst. Henri Poincaré Probab. Stat.*, 44:544–573, 2008.

- I. Benjamini, G. Kalai, and O. Schramm. First passage percolation has sublinear distance variance. *Ann. Probab.*, 31:1970–1978, 2003.
- J. van den Berg. A counterexample to a conjecture of J. M. Hammersley and D. J. A. Welsh concerning first-passage percolation. *Adv. in Appl. Probab.*, 15:465–467, 1983.
- S. Chatterjee and P. Dey. Central limit theorem for first-passage percolation time across thin cylinders. Available at <http://arxiv.org/pdf/0911.5702>, 2009.
- J. T. Cox. The time constant of first-passage percolation on the square lattice. *Adv. in Appl. Probab.*, 12:864–879, 1980.
- J. T. Cox and R. Durrett. Some limit theorems for percolation processes with necessary and sufficient conditions. *Ann. Probab.*, 9:583–603, 1981.
- J. T. Cox and H. Kesten. On the continuity of the time constant of first-passage percolation. *J. Appl. Probab.*, 18:809–819, 1981.
- R. Durrett. *Probability: Theory and examples*. Brooks/Cole [Thomson Learning], Belmont, third edition, 2005.
- O. Garot and R. Marchand. Asymptotic shape for the chemical distance and first-passage percolation on the infinite Bernoulli cluster. *ESAIM Probab. Stat.*, 8:169–199, 2004.
- O. Garot and R. Marchand. Large deviations for the chemical distance in supercritical Bernoulli percolation. *Ann. Probab.*, 35:833–866, 2007.
- J.-B. Gouréré. Shape of territories in some competing growth models. *Ann. Appl. Probab.*, 7:1273–1305, 2007.
- A. Gut. *Probability: A graduate course*. Springer-Verlag, New York, first edition, 2005.
- A. Gut. *Stopped random walks. Limit theorems and applications*. Springer, New York, second edition, 2009.
- J. M. Hammersley and D. J. A. Welsh. First-passage percolation, subadditive processes, stochastic networks and generalized renewal theory. In J. Neyman and L. Le Cam, editors, *Bernoulli, Bayes, Laplace Anniversary Volume*, Proc. Internat. Res. Semin., Statist. Lab., Univ. California, Berkeley, Calif., pages 61–110. Springer-Verlag, New York, 1965.
- C. Hoffman. Geodesics in first passage percolation. *Ann. Appl. Probab.*, 18:1944–1969, 2008.
- C. D. Howard. Models of first-passage percolation. In H. Kesten, editor, *Probability on discrete structures*, volume 110 of *Encyclopaedia Math. Sci.*, pages 125–173. Springer, Berlin, 2004.
- H. Kesten. Aspects of first-passage percolation. In *École d’Été de Probabilités de Saint Flour XIV - 1984*, volume 1180 of *Lecture Notes in Math.*, pages 125–264. Springer, Berlin, 1986.
- H. Kesten. On the speed of convergence in first-passage percolation. *Ann. Appl. Probab.*, 3:296–338, 1993.

- H. Kesten and Y. Zhang. A central limit theorem for "critical" first-passage percolation in two dimensions. *Probab. Theory Related Fields*, 107:137–160, 1997.
- T. Lindvall. *Lectures on the coupling method*. Dover, Mineola, 2002.
- C. M. Newman and M. S. T. Piza. Divergence of shape fluctuations in two dimensions. *Ann. Probab.*, 23:977–1005, 1995.
- R. Pemantle and Y. Peres. Planar first-passage percolation times are not tight. In *Probability and phase transition*, pages 261–264. Kluwer, Dordrecht, 1994.
- D. Richardson. Random growth in a tessellation. *Proc. Cambridge Philos. Soc.*, 74:515–528, 1973.
- H. Thorisson. *Coupling, stationarity, and regeneration*. Springer-Verlag, New York, 2000.
- Y. Zhang and Y. Zhang. A limit theorem for n_{0n}/n in first-passage percolation. *Ann. Probab.*, 12:1068–1076, 1984.